Lamb Shift of Unruh Detector Levels as a Probe of the Inflationary Vacuum

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Introduction

The Issue

- In de Sitter space, the particle scpectra inferred from the response of an Unruh detector and from the stress-energy tensor disagree.
- Explain why and how the detector nonetheless knows about the energy density.

Scalar Field in Expanding Background

Some Basics and Conventions

- Spatially flat, homogeneous background in conformal coordinates: $g_{\mu\nu} = a^2(\eta)\mathrm{diag}(1,-1,-1,-1)$

  De Sitter space: $a(\eta) = -\frac{1}{H\eta}, \quad \eta \epsilon]-\infty, 0[$

- $\sqrt{-g}\mathcal{L} = \sqrt{-g}\left(\frac{1}{2}g_{\mu\nu}\partial^\mu\phi\partial^\nu\phi - \frac{1}{2}m^2\phi^2\right)$

- Klein-Gordon equation for mode with comoving momentum k:
  $\left(\partial_\eta^2 + \left(\mathbf{k}^2 + a^2m^2\right) - \frac{a''}{a}\right)\varphi(\mathbf{k},\eta) = 0\,, \quad \varphi = a\phi\,, \quad ' \equiv d/d\eta$

  Solution for $m=0$: $\varphi(\mathbf{k},\eta) = \frac{1}{\sqrt{2k}}\left(1 - \frac{\mathrm{i}}{k\eta}\right)\mathrm{e}^{-\mathrm{i}k\eta}$

Particle Production: Mode Mixing

Parker 1969

Field operator

$$\varphi(x) = \int \frac{d^3k}{(2\pi)^3}\left(e^{i\mathbf{k}\cdot\mathbf{x}}\varphi(\mathbf{k},\eta)a(\mathbf{k}) + e^{-i\mathbf{k}\cdot\mathbf{x}}\varphi^*(\mathbf{k},\eta)a^\dagger(\mathbf{k})\right)$$

Nondiagonal Hamiltonian

$$H(\eta) = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\{\Omega(\mathbf{k},\eta)(a(\mathbf{k})a^\dagger(\mathbf{k})+a^\dagger(\mathbf{k})a(\mathbf{k}))+(\Lambda(\mathbf{k},\eta)a(\mathbf{k})a(-\mathbf{k})+\text{h.c.})\}$$

where

$$\begin{aligned}
\Omega(\mathbf{k},\eta) &= |\varphi'(\mathbf{k},\eta) - (a'/a)\varphi(\mathbf{k},\eta)|^2 + \omega^2(\mathbf{k},\eta)\,|\varphi(\mathbf{k},\eta)|^2 \\
\Lambda(\mathbf{k},\eta) &= \left(\varphi'(\mathbf{k},\eta) - \frac{a'}{a}\varphi(\mathbf{k},\eta)\right)^2 + \omega^2(\mathbf{k},\eta)\varphi^2(\mathbf{k},\eta) \\
\omega^2 &= \mathbf{k}^2 + m^2
\end{aligned}$$

Mode Mixing

- Bogolyubov transformation

$$\begin{pmatrix} \hat{a}(\mathbf{k}) \\ \hat{a}^\dagger(-\mathbf{k}) \end{pmatrix} = \begin{pmatrix} \alpha(k) & \beta^*(k) \\ \beta(k) & \alpha^*(k) \end{pmatrix} \begin{pmatrix} a(\mathbf{k}) \\ a^\dagger(-\mathbf{k}) \end{pmatrix}$$

with the norm $|\alpha(k)|^2 - |\beta(k)|^2 = 1$
Diagonal Hamiltonian

$$H(\eta) = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\omega(\mathbf{k},\eta)(\hat{a}(\mathbf{k})\hat{a}^\dagger(\mathbf{k}) + \hat{a}^\dagger(\mathbf{k})\hat{a}(\mathbf{k}))$$

$$n(\mathbf{k}) = \langle 0|\hat{a}^\dagger(\mathbf{k})\hat{a}(\mathbf{k})|0\rangle = |\beta(k)|^2 = \frac{\Omega(\mathbf{k})}{2\omega(\mathbf{k})} - \frac{1}{2} = a^2\left(\frac{H}{2k}\right)^2$$

- Intuitively expected result: mode energy density divided by the individual particle energy, minus the vacuum contribution

Adiabatic Expansion

Limited Applicability of Mode Mixing Picture

Note: β nonetheless oscillates $\propto \mathrm{e}^{2ik\eta}$

Also true for more general cases where for the WKB ansatz

$$\varphi(\mathbf{k},\eta)=\alpha(\mathbf{k})\left(2W(\mathbf{k},\eta)\right)^{-\frac{1}{2}}\mathrm{e}^{-\mathrm{i}\int^{\eta}d\eta' W(\mathbf{k},\eta')}+\beta(\mathbf{k})\left(2W(\mathbf{k},\eta)\right)^{-\frac{1}{2}}\mathrm{e}^{\mathrm{i}\int^{\eta}d\eta' W(\mathbf{k},\eta')}$$

one can adiabatically expand

$$\begin{aligned} {W^{(0)}}^2 &= \omega^2 \\ {W^{(2)}}^2 &= \omega^2 - (1-6\xi)\frac{a''}{a} + \frac{3}{4}\frac{{W^{(0)}}'^2}{{W^{(0)}}^2} - \frac{1}{2}\frac{{W^{(0)}}''}{W^{(0)}} \\ &\dots \end{aligned}$$

Energy Density in de Sitter Space

- Hamiltonian mode energy density

$$\Omega(\mathbf{k},\eta) = k + \frac{1}{2k\eta^2}$$

- Energy density component from the stress-energy tensor

$$\varrho = \langle 0|T^0{}_0(x)|0\rangle = \frac{1}{a^4}\int \frac{d^3k}{(2\pi)^3}\left(k + \frac{1}{2k\eta^2}\right)$$

- In addition to the quartic divergence of the cosmological term, there is a square divergence, power law behaviour.
- The square divergence can in principle be absorbed within Newton's constant G.
  However, not clear how this may fit into renormalizing gravity.

The Unruh Detector

Unruh 1976

- An idealized device only characterized by its energy levels, travels along trajectory $x(\tau)$ with proper time τ, couples to scalar field *via* $\hat{h}\phi$
- For simplicity assume that there are only two energy levels E_1 and E_2. Level spacing $\Delta E = E_2 - E_1$, define $h = \langle E_2|\hat{h}|E_1\rangle$

- $P(\tau)$ denotes probability for transition $E_1 \longrightarrow E_2$ after time τ has elapsed.
- Define $\mathcal{F}(\tau) = P(\tau)/|h|^2$.
  Applying quantum mechanical rules of time-dependent perturbation theory gives *response function*:

$$\frac{d\mathcal{F}(\Delta E)}{d\tau} = \int_{-\infty}^{\infty} d\Delta\tau e^{\mathrm{i}\Delta E \Delta\tau} \langle i|\phi\left(x(-\Delta\tau/2)\right)\phi\left(x(\Delta\tau/2)\right)|i\rangle$$

Test: Detector in Flat Space

- Consider Minkowski space filled with $\nu(|\mathbf{k}|)$ particles per mode
- Can be described in mode-mixing picture by

$$\varphi(\mathbf{k},t) = \frac{1}{\sqrt{2\omega(\mathbf{k})}} \left(\alpha(\mathbf{k})\mathrm{e}^{-\mathrm{i}\omega(\mathbf{k})t} + \beta(\mathbf{k})\mathrm{e}^{\mathrm{i}\omega(\mathbf{k})t}\right)$$

  with $|\beta(k)|^2 = \nu(|\mathbf{k}|)$, $|\alpha(k)|^2 = \nu(|\mathbf{k}|) + 1$ and β is constant in τ
- Take infinite time limit $t \to \infty$ for the response function

$$\frac{d\mathcal{F}_{\text{flat}}(\Delta E)}{dt} = \frac{k_{\Delta E}}{2\pi} \left[\nu(k_{\Delta E})\vartheta(\Delta E) + (\nu(k_{\Delta E}) + 1)\,\vartheta(-\Delta E)\right]$$

  with $k_{\Delta E} \equiv \sqrt{(\Delta E)^2 - m^2}$
- First term in square brackets: particle absorption
  Second term: spontaneous and induced emission

Response in de Sitter Space

Gibbons and Hawking '77, Higuchi '86, BG and Prokopec '04

- Freely falling detector

$$\frac{d\mathcal{F}(\Delta E)}{d\tau} = \frac{\Delta E}{2\pi}\left(1 + \frac{H^2}{\Delta E^2}\right)\frac{1}{\mathrm{e}^{(2\pi/H)\Delta E} - 1} \quad \text{for} \quad \Delta E \neq 0$$

- Indicates an exponentially falling spectrum of particles.
- Apparent contradiction with Parker's result or the energy density from $T^{\mu\nu}$.
- Reason: Mode mixing picture is inappropriate, oscillating Bogolyubov coefficient β

The Principle of Detailed Balance

Detector in equilibrium:

$$\mathcal{R}(E_1 \to E_2) = \mathcal{R}(E_2 \to E_1)$$

The response function fulfills the relation

$$\frac{d\mathcal{F}(\Delta E)}{d\tau} = \mathrm{e}^{-\beta\Delta E} \frac{d\mathcal{F}(-\Delta E)}{d\tau}$$

Introduce occupation numbers $n(E_1)$ and $n(E_2)$

$$n(E_1)\frac{dP(E_1 \to E_2)}{d\tau}(1 + n(E_2)) = n(E_2)\frac{dP(E_2 \to E_1)}{d\tau}(1 + n(E_1))$$

$$\Longrightarrow$$

$$n(E_2) = \frac{1}{\mathrm{e}^{\beta\Delta E} - 1}$$

Is the Unruh Detector Blind to the Energy Density?

Question

What is the significance of the power-law behaviour of the energy density?

Lamb Shift

What Happens to the Vacuum

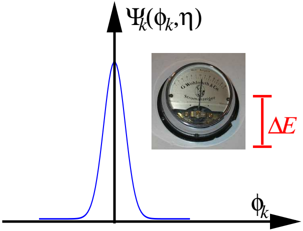
- Mode mixing picture is not appropriate to account for "particle production" in the expanding background.
- However, the amplitude of the modes grow.
- In the functional picture, this corresponds to a growth of the vacuum fluctuations, *cf.* the generation of cosmic perturbations.

And how the Detector Knows about it

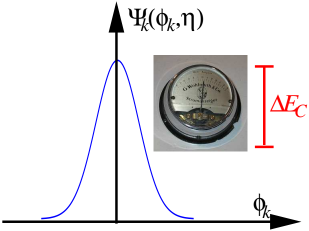
- Energy levels of an atom are sensitive to vacuum fluctuations. The Lamb shift gives a correction at one loop order (a keystone success of QED, `H. Bethe 1947`).
- We are agnostic about the detector's inner structure, however we can think about it as a bound state with discrete energy levels.
- Energy levels acquire corrections by vacuum fluctuations.

Lamb Shift in Curved Spacetime

- The detector's ΔE is defined in flat space. This already includes the infinite correction δE_M due to Lamb shift.
- However, we can *compare* a detector in flat and in curved space.



ΔE in flat space already includes the Lamb-shift renormalization.



Vacuum fluctuations grow due to spacetime expansion. Lamb shift yields a different contribution. $\Longrightarrow \Delta E_C \neq \Delta E$

- We can observe $\delta E = \delta E_C - \delta E_M$, note that δE is finite.

Calculation of Lamb Shift

2nd Order Perturbation Theory

$$\delta E_X = \int \frac{d^3k}{(2\pi)^3} \frac{\left| \int \frac{d^3k'}{(2\pi)^3} \langle \mathbf{k}', E_2 | \hat{h} a^\dagger(\mathbf{k}) \varphi(\mathbf{k}, \eta) | 0, E_1 \rangle \right|^2}{\Delta E - \Omega(\mathbf{k})}$$

$$= \int \frac{d^3k}{(2\pi)^3} \frac{\left| h^2_{mn} \right| \left| \varphi(\mathbf{k}, \eta) \right|^2}{\Delta E - \Omega(\mathbf{k})}$$

In Minkowski Space

$$\delta E_{\mathrm{M}}^{m=0} = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2k} \frac{h^2}{\Delta E - k} = \frac{h^2}{4\pi^2} \left[-k - \Delta E \log |\Delta E - k| \right]_0^\infty$$

Lamb Shift in de Sitter Space

$$\begin{aligned}
\delta E_{\rm dS}^{m=0} &= \int \frac{d^3k}{(2\pi)^3}\left(\frac{1}{2k}+\frac{H^2}{2k^3}\right)\frac{h^2}{\Delta E-\left(k+\frac{H^2}{k}\right)} \\
&= \frac{h^2}{4\pi^2}\left[-k+\frac{\Delta E^2/4}{\sqrt{\Delta E^2/4-H^2}}\log\left|\frac{k-\Delta E/2+\sqrt{\Delta E^2/4-H^2}}{k-\Delta E/2-\sqrt{\Delta E^2/4-H^2}}\right|\right. \\
&\qquad \left.-\frac{\Delta E}{2}\log\left|\frac{(k+\Delta E/2)^2}{\Delta E^2/4-H^2}-1\right|\right]_0^\infty
\end{aligned}$$

The observable difference is finite:

$$\begin{aligned}
\delta E &= \delta E_{\rm dS}^{m=0}-\delta E_{\rm M}^{m=0} \\
&= \frac{h^2}{4\pi^2}\left\{\Delta E\log\left|\frac{H}{\Delta E}\right|-\frac{\Delta E^2}{4\sqrt{\Delta E^2/4-H^2}}\log\left|\frac{\Delta E/2-\sqrt{\Delta E^2/4-H^2}}{\Delta E/2+\sqrt{\Delta E^2/4-H^2}}\right|\right\}
\end{aligned}$$

Lamb Shift in de Sitter Space

... and condenses considerably when expanded in $H/\Delta E$

$$\delta E = \frac{h^2}{4\pi^2}\frac{H^2}{\Delta E}\left(-1 - 2\log\left|\frac{H}{\Delta E}\right| + O\left(\frac{H}{\Delta E}\right)\right)$$

Remarks

- Both, the amplitude $|\varphi(\mathbf{k})|$ and the mode energy $\Omega(\mathbf{k})$ contribute to the Lamb shift.
- Lamb shift corresponds to a mixing of unperturbed detector levels
  $\Longrightarrow$ Can quantitatively compare with the detector in equilibrium and the occupation numbers
- Unruh detector sees the power-law behaviour, Lamb shift is more important in the UV.

Remarks

- Similar behaviour for massive scalar in general FLRW-spacetimes:

$$\begin{aligned}\delta E &= \delta E_{\mathrm{FLRW}} - \delta E_{\mathrm{M}} \\ &= \frac{h^2}{4\pi^2}\left\{-\frac{5}{12}\frac{1}{\Delta E}\frac{a''}{a} - \frac{1}{2}\frac{1}{\Delta E}\frac{a'^2}{a^2} + \frac{1-6\xi}{2}\frac{1}{\Delta E}\log\frac{2\Delta E}{m}\frac{a''}{a}\right. \\ &\qquad \left. -\frac{3\pi}{16}\frac{m}{\Delta E^2}\frac{a''}{a} - \frac{3\pi}{32}\frac{m}{\Delta E^2}\frac{a'^2}{a^2} + O\left(\frac{m^2}{\Delta E^3}\right)\right\}\end{aligned}$$

- It is not clear whether the expression for the response function is correct. LSZ reduction applicable?
- Apparently related to particle self energies in de Sitter, which are $\propto H$ rather than $\propto \exp(-H/\mu)$, where μ is some mass scale.

Lamb Shift in Rindler Space

Consider accelerated observer in $D = 2$ on trajectory x

- Invariant acceleration: $\alpha = \left[(d^2x/d\tau^2)^2\right]^{\frac{1}{2}}$
- Mode amplitude (λ corresponds to $\sqrt{k^2 + m^2}$)

$$|\varphi_\lambda(\xi = 0, \tau)|^2 = \frac{1}{2\lambda} \frac{1 + \frac{1}{2}\frac{m^2}{\lambda^2} + \frac{3}{8}\frac{m^4}{\lambda^4} + \dots}{1 - \mathrm{e}^{-2\pi|\lambda|/\alpha}} \left(1 + \frac{1}{2}\frac{\alpha^2 m^2}{\lambda^4} + \dots\right)$$

- Local, virtual mode energy

$$\Omega_\lambda = \frac{1}{|\lambda|}\left(\lambda^2 - \frac{3}{8}\frac{\alpha^2 m^4}{\lambda^4} + \dots\right)$$

- Lamb shift

$$\delta E = \delta E_R - \delta E_M = \frac{h^2}{6\pi}\frac{\alpha^2}{\Delta E m^2} + \begin{cases} \frac{h^2}{8\pi}\frac{\alpha}{m\Delta E} & \text{for } m \ll \alpha \\ \frac{h^2}{4\pi\Delta E}\sqrt{\frac{\alpha}{m}}\mathrm{e}^{-\frac{2\pi m}{\alpha}} & \text{for } M \gg \alpha \end{cases}$$

Conclusions

- The process referred to as “particle production” in the expaning Universe yields a power-law spectrum (not exponentially falling!) for the energy density.
- This does however *not* correspond to the presence of particles, since it is not captured in the response rate of a detector.
- The effect becomes however manifest in the Lamb shift of energy levels of the detector.
- The expanding background in first place alters self energy corrections rather than producing “particles”.