Non-linear Inflationary Perturbations on Long Wavelengths

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Linear theory leads to the standard statement: "Inflation produces Gaussian Perturbations".

However:

- Gravity is non-linear. Some non-Gaussianity will always be present.
- Precision Cosmology: Non-Linearities may be observable.
- Potentially useful for further testing inflation.

A methodology for doing non-linear calculations in general inflationary models is needed.

Long Wavelengths

Focus on scales larger than $R \simeq \frac{c}{aH}$ (Hubble Radius)

- Approximation: Drop second order spatial gradients

$$ds^2 = -N^2(t, \mathbf{x})dt^2 + e^{2\alpha}(t, \mathbf{x})h_{ij}(\mathbf{x})dx^i dx^j$$

General scalar E.M. tensor:

$$T_{\mu\nu} = G_{AB}\partial_\mu\phi^A\partial_\nu\phi^B - g_{\mu\nu}\left(\frac{1}{2}G_{AB}\partial^\lambda\phi^A\partial_\lambda\phi^B + V\right)$$

$$\Pi^A \equiv \frac{\dot{\phi}^A}{N}, \quad H \equiv \frac{\dot{a}}{Na}$$

$\Rightarrow$ "Separate Universe Picture"

Separate Universes

- Constraints:

$$
\begin{aligned}
H^2 &= \frac{8\pi}{3m_{\rm pl}^2}\left(\frac{1}{2}\Pi_B\Pi^B + V\right) \\
\partial_i H &= -\frac{4\pi}{m_{\rm pl}^2}\Pi_B\partial_i\phi^B
\end{aligned}
$$

- Evolution equations:

$$
\begin{aligned}
\frac{dH}{dt} &= -\frac{4\pi}{m_{\rm pl}^2}N\Pi_B\Pi^B \\
\mathcal{D}_t\Pi^A &= -3NH\Pi^A - NG^{AB}V_B
\end{aligned}
$$

In separate universe picture spatial gradients conveniently characterize inhomogeneity

Equations of Motion

Define a non-linear variable: $\zeta_i^A = -\frac{\kappa}{\sqrt{2\tilde{\epsilon}}}\left(\partial_i\phi^A - \frac{\Pi^A}{H}\partial_i\alpha\right)$,
$\left(\tilde{\epsilon} \equiv \frac{\kappa^2}{2}\frac{\Pi^2}{H^2}\right)$

- The ζ_i^A describe the inhomogeneous spacetime completely
- The long wavelength dynamics can be expressed in terms of ζ_i^A

From the Einstein equations one gets:

$$\boxed{\frac{\mathcal{D}^2}{dt^2}\zeta_i^A - F(\phi, \Pi, H)\frac{\mathcal{D}}{dt}\zeta_i^A + M^A{}_B(\phi, \Pi, H)\zeta_i^B = 0}$$

$$\partial_i\phi^A = I^A{}_B\zeta_i^B + \tilde{I}^A\partial_i\alpha, , \qquad \partial_i H = J_B\zeta_i^B + \tilde{J}^A\partial_i\alpha$$

$$\partial_i\Pi^A = K^A{}_B\zeta_i^B + L^A{}_B\dot{\zeta}_i^B + \tilde{L}^A\partial_i\alpha$$

Where $I^A{}_B = I^A{}_B(\phi, \Pi, H)$ e.t.c.

Some properties of ζ_i^A

- $\zeta_i^A = \partial_i \zeta_{lin} + \dots$
- ζ_i^A is invariant under changes of time slicing
- ζ_i is exactly conserved in the single field case

Its e.o.m.

- Is exact under the long wavelength approximation
- Is formally the same as that of gauge invariant linear theory and reproduces it to linear order
- A perturbative expansion to second order is relatively simple and transparent
- Suggests a connection with short scales

A useful gauge

In a perturbed inflationary universe it is natural to choose as time variable:

$$t = \ln aH \Leftrightarrow N = \frac{1}{H(1-\tilde{\epsilon})}$$

- All modes exit the horizon and enter the long wavelength system simultaneously
- the equations are simplified
- connection with short scales easier

Linear Perturbations

$$\delta q^A = a\left(\delta\phi^A - \frac{\dot{\phi}^A}{H}\Psi\right)$$

$$\delta\hat{q}^A(t,\mathbf{x}) =$$

$$\int \frac{d^3k}{(2\pi)^{3/2}} \sum_B \left[Q^A{}_B(k)\hat{a}_B(\mathbf{k})e^{i\mathbf{kx}} + Q^{*A}{}_B(k)\hat{a}^\dagger_B(\mathbf{k})e^{-i\mathbf{kx}} \right]$$

$$\frac{\mathcal{D}^2}{dt^2}Q^A{}_B - \left(\frac{\dot{N}}{N} - NH\right)\frac{\mathcal{D}}{dt}Q^A{}_B + \left(\left(\frac{Nk}{a}\right)^2 \delta^A{}_C + \Omega^A{}_C \right) Q^C{}_B = 0$$

On long wavelengths consider:

$$\delta q^A(t,\mathbf{x}) =$$

$$\int \frac{d^3k}{(2\pi)^{3/2}} \sum_B \frac{1}{\sqrt{2}} \left(Q^A{}_B(k)\alpha^B(\mathbf{k}) + Q^{A*}{}_B(k)\alpha^{B*}(-\mathbf{k}) \right) e^{i\mathbf{kx}}$$ with α^A

complex random numbers: $\langle \alpha^A(\mathbf{k})\, \alpha^{B*}(\mathbf{k}')\rangle = (2\pi)^{3/2}\delta^{AB}\delta(\mathbf{k}-\mathbf{k}')$

$\langle \alpha^A(\mathbf{k})\, \alpha^B(\mathbf{k}')\rangle = 0$

- Smooth with a window function:
  $W(|\mathbf{x}-\mathbf{x}'|/R) = \frac{1}{(2\pi)^{3/2}R^3} e^{-|\mathbf{x}-\mathbf{x}'|^2/2R^2}$
  $\delta\bar{q}^A(\mathbf{x}) = \int d^3x' \delta q^A(\mathbf{x}') W\left(\frac{|\mathbf{x}-\mathbf{x}'|}{R}\right)$
- The equation of motion for the smoothed field is:
  $\frac{\mathcal{D}}{dt}\bar{\delta q}^A - \bar{\delta\theta}^A = \int \frac{d^3k}{(2\pi)^{3/2}} \delta q^A(\mathbf{k}) \dot{\mathcal{W}}(k) e^{i\mathbf{kx}} + c.c.$
  $\frac{\mathcal{D}}{dt}\bar{\delta\theta}^A + (\ldots)\bar{\delta\theta}^A + (\ldots)^A{}_B \bar{\delta q}^B = \int \frac{d^3k}{(2\pi)^{3/2}} \delta\theta^A(\mathbf{k}) \dot{\mathcal{W}}(k) e^{i\mathbf{kx}} + c.c.$
- With δq^A, $\delta\theta^A$ random fields, the equations of motion for the long wavelength linear variables become stochastic.
- Knowledge of the linear solutions up to horizon crossing determines the source terms on the r.h.s.

Of course, for linear theory this formulation is redundant.

Stochastic Equations for Long Wavelengths

However, the l.h.s. *with the coefficients made spatially dependent* is an *exact* non-linear long wavelength equation for $\mathcal{Q}_i^A = -\frac{a\sqrt{2\tilde{\epsilon}}}{\kappa}\zeta_i^A = e^\alpha\left(\partial_i\phi^A - \frac{\Pi^A}{H}\partial_i\alpha\right)$.

Therefore we postulate:

$$\begin{cases} \mathcal{D}_t\zeta_i^A - \theta_i^A = \mathcal{S}_i^A \\ \mathcal{D}_t\theta_i^A + \dfrac{3-2\tilde{\epsilon}+2\tilde{\eta}^\parallel - 3\tilde{\epsilon}^2 - 4\tilde{\epsilon}\tilde{\eta}^\parallel}{(1-\tilde{\epsilon})^2}\,\theta_i^A + \dfrac{1}{(1-\tilde{\epsilon})^2}\,\Xi^A{}_B\,\zeta_i^B = \mathcal{J}_i^A \end{cases}$$

with

$$\mathcal{S}_i^A \equiv \frac{-\kappa}{2a\sqrt{\tilde{\epsilon}}}\int\frac{d^3k}{(2\pi)^{3/2}}\,\dot{\mathcal{W}}(k)\,Q_{\mathrm{lin}\,B}^A(k)\alpha^B(\mathbf{k})\,\mathrm{i}k_i\,\mathrm{e}^{\mathrm{i}\mathbf{kx}} + \mathrm{c.c.}$$

$$\mathcal{J}_i^A \equiv \frac{-\kappa}{2a\sqrt{\tilde{\epsilon}}}\int\frac{d^3k}{(2\pi)^{3/2}}\,\dot{\mathcal{W}}(k)\,\left[\mathcal{D}_tQ_{\mathrm{lin}\,B}^A(k),Q_{\mathrm{lin}\,B}^A\right]\alpha^B(\mathbf{k})\,\mathrm{i}k_i\,\mathrm{e}^{\mathrm{i}\mathbf{kx}} + \mathrm{c.c.}$$

We have defined:

$$\tilde{\epsilon}(t,\mathbf{x}) \equiv \frac{\kappa^2}{2}\frac{\Pi^2}{H^2}\,, \quad \tilde{\eta}^A(t,\mathbf{x}) \equiv -\frac{3H\Pi^A + G^{AB}\partial_B V}{H\Pi}, \quad \tilde{\eta}^{\parallel} \equiv \frac{\Pi_A}{\Pi}\tilde{\eta}^A$$

and $\Xi^A{}_B$ depends on two more "slow roll" parameters:

$$\left(\tilde{\eta}^{\perp}\right)^2(t,\mathbf{x}) \equiv \frac{V^A V_A - (\frac{\Pi^A}{\Pi}V_A)^2}{H^2\Pi^2}\,, \quad \tilde{\xi}^{\parallel}(t,\mathbf{x}) \equiv 3\tilde{\epsilon} - 3\tilde{\eta}^{\parallel} - \frac{\frac{\Pi^A}{\Pi}\frac{\Pi^B}{\Pi}V_{AB}}{H^2}.$$

Initial conditions:

$$\lim_{t\to-\infty}\zeta_i^A = 0\,, \quad \lim_{t\to-\infty}\theta_i^A = 0$$

Constraints

$$\partial_i \alpha = -\partial_i (\ln H) = -\frac{\tilde{\epsilon}}{1-\tilde{\epsilon}} \frac{\Pi_B}{\Pi} \zeta_i^B \,,$$

$$\partial_i \phi^A = -\frac{\sqrt{2\tilde{\epsilon}}}{\kappa} \left(\delta^A_B - \frac{\tilde{\epsilon}}{1-\tilde{\epsilon}} \frac{\Pi^A}{\Pi} \frac{\Pi_B}{\Pi} \right) \zeta_i^B \,,$$

$$\begin{aligned} \mathcal{D}_i \Pi^A \quad = \quad & -\frac{\sqrt{2\tilde{\epsilon}}}{\kappa} H \Bigg[(1-\tilde{\epsilon})\, \theta_i^A + \Bigg(\left(\tilde{\epsilon} + \tilde{\eta}^{\parallel} \right) \delta^A_B \\ & -\tilde{\epsilon} \frac{\Pi^A}{\Pi} \frac{\Pi_B}{\Pi} + \frac{\tilde{\epsilon}}{1-\tilde{\epsilon}} \tilde{\eta}^A \frac{\Pi_B}{\Pi} \Bigg) \zeta_i^B \Bigg] \,. \end{aligned}$$

The system of equations closes and is self consistent.

Conclusions

A stochastic framework:

- Non-linear evolution for gauge invariant variables
- Valid for multi-field inflationary models
- Includes metric perturbations
- No slow-roll assumption required
- Allows for a relatively simple investigation of non-Gaussianity during inflation $\Rightarrow$ analytic, semi-analytic approach & numerical simulations

See Bartjan's talk for some interesting results...!