Non-linear cosmological perturbations:

evolution and conservation

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Why going non-linear

Linear theory describes remarkably well perturbations in the universe

$$\frac{\delta T}{T} \sim 10^{-5} \quad \Rightarrow \quad \delta g_{\mu\nu} \sim 10^{-5}$$

Linear cosmological perturbations is an excellent approx.

Non-linear aspects:

- *Inhomogeneities on scales larger than Hubble scale* H^{-1}
- *Backreaction of non-linear perturbations on the background universe*
- *Non-Gaussianities*

$$\delta(t) = a(t)\delta + b(t)\delta^2$$

$\Rightarrow$ Second order cosmological perturbations

$\Rightarrow$ Fully non-linear approach:

- *covariant and non-perturbative formalism*
- *evolution and conservation of non-linear perturbations at all scales*

Coordinate approach and gauge invariance

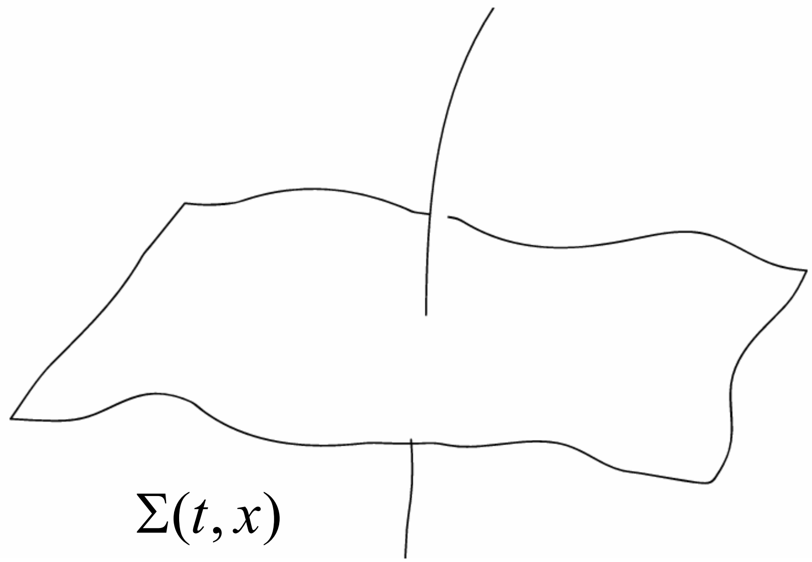
There exists an ideal smooth universe (background). Perturbations are defined with respect to it

[Bardeen, '82]

Energy density: $\delta\rho(t,x)=\rho(t,x)-\bar{\rho}(t)$

Metric: $\delta g_{\mu\nu}(t,x)=g_{\mu\nu}(t,x)-\bar{g}_{\mu\nu}(t)$

Splitting meaningful only with respect to a given coordinate system



- *Gauge transformation:*

  change in the correspondence between the perturbed and background universe

- *Gauge invariant quantities:*

  combination of gauge-dependent quantities invariant under gauge transformation

- *Physical and geometrical meaning:*

  definition on a hypersurface

Curvature perturbation on uniform energy hypersurfaces ζ

- *Perturbed metric:*

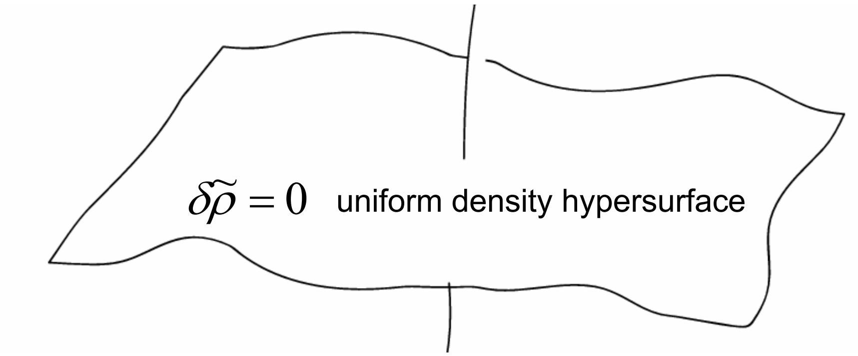
$$ds^2 = a^2\{-(1+2A)d\eta^2 + 2\partial_i B dx^i d\eta + [(1-2\psi)\delta_{ij} + 2\partial_i\partial_j E]dx^i dx^j\}$$

- *Gauge transformation:*

$$\eta \to \tilde{\eta} = \eta + \varepsilon$$

$$\delta\rho \to \delta\tilde{\rho} = \delta\rho + \varepsilon\dot{\rho}$$

$$\psi \to \tilde{\psi} = \psi - H\varepsilon$$



- *Gauge invariant quantity: curvature perturbation on the uniform density hypersurface*

$$\delta\tilde{\rho} = 0 \quad \Rightarrow \quad \underline{-\tilde{\psi} \equiv \zeta = -\psi - H\frac{\delta\rho}{\dot{\rho}}}$$

[Bardeen, Steinhardt, Turner]

Theorem [Wands, Malik, Lyth, Liddle]: Assuming only energy conservation

$$T^\nu_\mu = (\rho + P)u^\nu u_\mu + P g^\nu_\mu \qquad u^\mu \nabla_\nu T^\nu_\mu = 0$$

$$\boxed{\zeta' = -\frac{H}{(\rho+P)}\delta P_{\mathrm{nad}} - \frac{1}{3}\nabla^2(E' + v)}$$

$$\delta P_{\mathrm{nad}} = \delta P - c_s^2 \delta\rho$$

Non-adiabatic pressure

On large scales, ζ is conserved for adiabatic perturbations, $\zeta' \approx 0$

Covariant approach

Work only with geometrical quantities [Hawking and Ellis, 60'-70']

- Projector on the orthogonal hypersurface

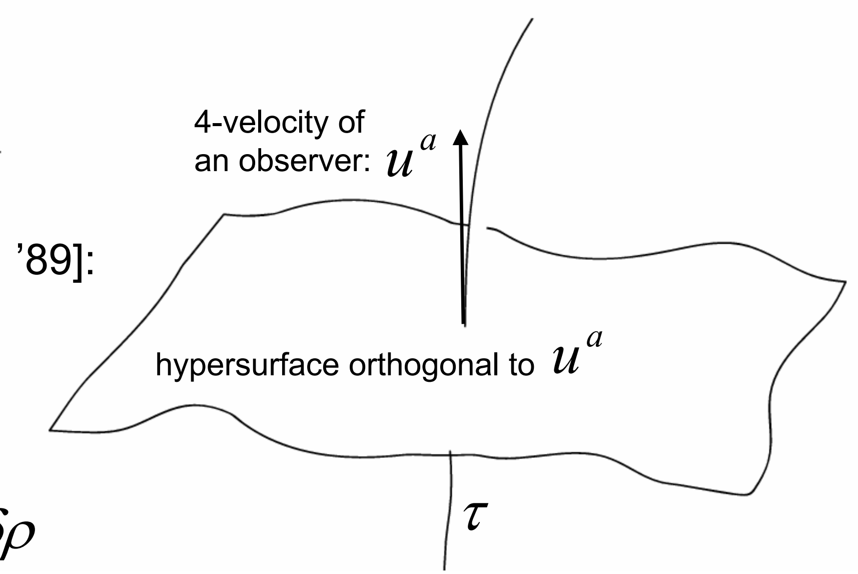
$$h^a_b = g^a_b + u^a u_b$$

- *How do we define perturbations?* [Bruni and Ellis, '89]:

projected gradient: $X_b = D_b \rho = h^a_b \nabla_a \rho$

- In a coordinate system:

$$X_i = \partial_i \rho(t,x) = \partial_i \delta\rho(t,x) + \partial_i \bar{\rho}(t) = \partial_i \delta\rho$$



Definitions:

$\Theta = \nabla_a u^a$ volume expansion $\qquad (\bar{\Theta} = 3H \quad \text{in FLRW})$

$\alpha = \frac{1}{3}\int \Theta d\tau$ integrated volume expansion along the worldline of the observer

$\Theta = 3\dot{\alpha} \quad \text{with} \quad \dot{\alpha} = u^a \nabla_a \alpha \qquad (e^{\bar{\alpha}} = a \quad \text{in FLRW})$

directional derivative of a scalar

$L_u X_a = u^c \nabla_c X_a + X_c \nabla_a u^c$ Lie derivative: change of X_a along u^a

directional derivative of a vector

Generalizing the conserved quantity ζ to non-linear order

Inspired by the work of Wands, Malik, Lyth, Liddle:

$$T^a_b = (\rho + P) u^a u_b + P g^a_b \qquad\qquad u^b \nabla_a T^a_b = 0 \quad \Rightarrow$$

1) Covariant and non-perturbative energy conservation equation:

$$\dot{\rho} + \Theta(\rho + P) = 0$$

2) Applying the spatially projected derivative:

$$D_a(\dot{\rho}) + D_a(3\dot{\alpha})(\rho + P) + \Theta(D_a\rho + D_a P) = 0$$

3) Inverting the spatial gradient and the time (Lie) derivative:

$$L_u(D_a\rho) + L_u(D_a\alpha)3(\rho + P) + \Theta(D_a\rho + D_a P) = 0$$

It is natural to introduce the quantity

$$\zeta_a \equiv D_a\alpha + \frac{D_a\rho}{3(\rho + P)}$$

Integrated expansion: local number of e-folds

$$\zeta_a \equiv D_a\alpha - \frac{\dot{\alpha}}{\dot{\rho}} D_a\rho$$

Non-perturbative generalization of ζ

$$\boxed{L_u\zeta_a = -\frac{\Theta}{3(\rho + P)}\Gamma_a}$$

$$\Gamma_a \equiv D_a P - \frac{\dot{P}}{\dot{\rho}} D_a\rho$$

Non-perturbative generalization of $\delta P_{\rm nad}$

Non-perturbative evolution equation*,*
valid at all scales and at all order in the perturbations

Recovering the linear theory

Non-linear covariant equation

$$L_u \zeta_a = -\frac{\Theta}{3(\rho+P)}\Gamma_a$$

$$\zeta_a = D_a\alpha - \frac{\dot{\alpha}}{\dot{\rho}} D_a\rho$$

$\longrightarrow$

Linear coordinate based equation

$$\zeta' = -\frac{H}{(\rho+P)}\delta P_{\mathrm{nad}} - \frac{1}{3}\nabla^2(E'+v)$$

$$\zeta = -\psi - H\frac{\delta\rho}{\rho'}$$

1) $$D_a\alpha = h_a^b \nabla_b \alpha = \partial_a \alpha + u_a u^b \nabla_b \alpha \quad \Rightarrow \quad \zeta_a = \partial_a\alpha - \frac{\dot{\alpha}}{\dot{\rho}}\partial_a \rho$$

2) $$\alpha(t,x) = \overline{\alpha}(t) + \delta\alpha^{(1)}(t,x)$$

3) $$\zeta_i^{(1)} = \partial_i\left(\delta\alpha^{(1)} - H\frac{\delta\rho^{(1)}}{\overline{\rho}'}\right)$$

4) $$3\dot{\alpha} = \Theta = \nabla_a u^a \quad \Rightarrow \quad \delta\alpha^{(1)} = -\psi + \frac{1}{3}\nabla^2\left(E + \int v\, d\eta\right)$$

5) $$L_u \zeta_i^{(1)} = u^c \partial_c \zeta_i^{(1)} + \zeta_c^{(1)} \partial_i u^c = \zeta_i^{(1)}{}' / a$$

Second order perturbations

- *Expand at 2nd order in the perturbations*:

[See Malik's talk]

$$\alpha(t,x) = \overline{\alpha}(t) + \delta\alpha^{(1)}(t,x) + \frac{1}{2}\delta\alpha^{(2)}(t,x)$$

- *Find automatically the gauge invariant quantity conserved at 2nd order:*

$$\zeta_i^{(2)} = \partial_i \zeta^{(2)} + \frac{2}{\overline{\rho}'}\delta\rho^{(1)}\zeta_i^{(1)}$$

with

$$\zeta^{(2)} = \delta\alpha^{(2)} - \frac{\overline{\alpha}'}{\overline{\rho}'}\delta\rho^{(2)} - \frac{2}{\overline{\rho}'}\delta\alpha^{(1)}{}'\delta\rho^{(1)} + 2\frac{\overline{\alpha}'}{\overline{\rho}'^2}\delta\rho^{(1)}{}'\delta\rho^{(1)} + \frac{1}{\overline{\rho}'}\left(\frac{\overline{\alpha}'}{\overline{\rho}'}\right)'\delta\rho^{(1)2}$$

- *Conservation equation for the 2nd order* ζ *variable at all scales*

$$\zeta^{(2)}{}' = -\frac{H}{\overline{\rho}+\overline{P}}\Gamma^{(2)} - 2\frac{H}{\overline{\rho}+\overline{P}}\Gamma^{(1)}\zeta^{(1)}{}' - 2v^i\partial_i\zeta^{(1)}$$

[Malik and Wands, '02]

Do not need to go at second order!
Simpler to work with covariant variables

Non-perturbative conservation equation: a conclusion

- *Covariant and geometrical variable*: describes deviations from FLRW universe at *any order* in perturbations (non-perturbative analog of curv. pert. on uniform density hyp.)

$$\zeta_a = D_a\alpha - \frac{\dot\alpha}{\dot\rho} D_a\rho \qquad S = \exp(\alpha)$$

local scale factor (physical quantity)

(separate universe approach)

- *Simple non-perturbative evolution equation, valid at all scales*

$$L_u\zeta_a = -\frac{\Theta}{3(\rho+P)}\Gamma_a$$

Can recover very easily the results of the literature at first and second order

- *Non-perturbative analog of curv. pert. on comoving hypersurfaces* (scalar fields)

$$L_u R_a = \frac{\Theta}{3(\rho+P)}\Gamma_a + \dots$$

- *Other non-linear developments in the literature*:

  [Rigopoulos, Shellard, '03] and [Lyth, Malik, Sasaki, '04] :

  coordinate-based (ADM), no small scale evolution