COSMO 05

Cosmological perturbations: recent developments at second order

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Overview

Scalar field dynamics using cosmological perturbation theory during inflation

- Short linear review
  - Gauge-invariant variables
  - Klein-Gordon equation
- Second order perturbation theory
  - Gauge-invariant variables
  - Klein-Gordon equation
  - Problem, and solutions
- Conclusions and Outlook

Other approaches: see talks by Lyth, Rigopoulos, van Tent, Vernizzi . . .

Linear perturbations: the metric

Scalar perturbations only (flat background)

$$g_{\mu\nu} = a(\eta)^2 \begin{pmatrix} -1 - 2\phi_1 & B_{1,i} \\ B_{1,j} & [(1 - 2\psi_1)\delta_{ij} + 2E_{1,ij}] \end{pmatrix}$$

Bardeen 1980

where ψ_1 is the curvature perturbation, related to the Ricci-scalar/intrinsic curvature of a spatial 3-hypersurface by

$$^{(3)}R_1 = \frac{4}{a^2}\nabla^2\psi_1$$

Notation: all background quantities functions of conformal time η only, all perturbations functions of x^μ, and $\mu, \nu = 0, \ldots, 3$, $i, j = 1, 2, 3$

Linear gauge-invariant quantities

A first order coordinate transformation $x^\mu \to \widetilde{x^\mu} = x^\mu + \delta x_1^\mu$, where $\delta x_1^\mu = [\delta\eta_1, \delta x_1^i]$, induces a change in the curvature perturbation

$$\widetilde{\psi}_1 = \psi_1 - \mathcal{H}\delta\eta_1$$

where $\mathcal{H} \equiv \frac{a'}{a}$ and, splitting the scalar field $\varphi(\eta, x^i) \equiv \varphi_0(\eta) + \delta\varphi_1(\eta, x^i)$, in the scalar field perturbation

$$\widetilde{\delta\varphi_1} = \delta\varphi_1 + \varphi_0'\delta\eta_1$$

Combine both and get gauge-invariant quantity,

$$\mathcal{Q}_1 \equiv \delta\varphi_1 + \frac{\varphi_0'}{\mathcal{H}}\psi_1$$

the “field perturbation on uniform curvature hypersurfaces”

Sasaki 1986, Mukhanov 1988

Scalar field dynamics

- Energy momentum tensor of minimally coupled scalar field with potential $U(\varphi)$

$$T_{\mu\nu} = \varphi_{,\mu}\varphi_{,\nu} - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} \varphi_{,\alpha}\varphi_{,\beta} - g_{\mu\nu} U(\varphi)$$

- energy-momentum conservation $\nabla_\mu T^{\mu\nu} = 0$ gives *Klein Gordon equation*
  - in the background

$$\varphi_0'' + 2\mathcal{H}\varphi_0' + a^2 U_{,\varphi} = 0$$

  - and to first order, on large scales,

$$\delta\varphi_1'' + 2\mathcal{H}\delta\varphi_1' + 2a^2 U_{,\varphi}\phi_1 - 3\varphi_0'\psi_1' - \varphi_0'\phi_1' + a^2 U_{,\varphi\varphi}\delta\varphi_1 = 0$$

Scalar field dynamics

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- Using first order Einstein equations (and background equations), get Klein-Gordon equations in terms of gauge-invariant variables

$$\mathcal{Q}_1'' + 2\mathcal{H}\mathcal{Q}_1' + \left[a^2 U_{,\varphi\varphi} - \frac{8\pi G}{a^2}\left(a^2\varphi_0'\left(\frac{\varphi_0'}{\mathcal{H}}\right)\right)'\right]\mathcal{Q}_1 = 0$$

Hwang 1996, Taruya and Nambu 1998, Gordon et al. 2000

- Relation between field fluctuations and ζ_1,

$$\zeta_1 = -\frac{\mathcal{H}}{\varphi_0'}\mathcal{Q}_1$$

where ζ_1 is the curvature perturbation on uniform density hypersurfaces

Can easily be extended to multiple fields . . .

So what. . .

So far first order theory only, but . . .

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So far first order theory only, but . . .

- second order perturbation theory is very similar to standard first order theory:
  - single field case follows directly,
  - multi-field case is slightly trickier,
- but both are more work.

Second order theory

Splitting tensors into homogeneous background and inhomogeneous perturbations, e.g. the metric tensor

$$g_{\mu\nu}(\eta, x^i) = {g_{\mu\nu}}^{(0)}(\eta) + \delta g^{(1)}_{\mu\nu}(\eta, x^i) + \frac{1}{2}\delta g^{(2)}_{\mu\nu}(\eta, x^i) + \ldots$$

Under coordinate transformation

$$\widetilde{x^\mu} = x^\mu + \xi_1^\mu + \frac{1}{2}\left(\xi^\mu_{1,\nu}{\xi_1}^\nu + \xi_2^\mu\right) + \ldots$$

perturbations transform as

$$\begin{aligned}
\widetilde{\delta g^{(1)}_{\mu\nu}} &= \delta g^{(1)}_{\mu\nu} + \pounds_{\xi_1} g^{(0)}_{\mu\nu} \\
\widetilde{\delta g^{(2)}_{\mu\nu}} &= \delta g^{(2)}_{\mu\nu} + \pounds_{\xi_2} g^{(0)}_{\mu\nu} + \pounds^2_{\xi_1} g^{(0)}_{\mu\nu} + 2\pounds_{\xi_1}\delta g^{(1)}_{\mu\nu}
\end{aligned}$$

Mukhanov, Abramo, and Brandenberger (1997)

Bruni, Matarrese, Mollerach and Sonego (1997)

Second order gauge invariant variables

- To simplify things: large scales only, no need to specify "threading".
- Curvature perturbation then changes under a second order coordinate transformations as

$$\widetilde{\psi_2} = \psi_2 - \delta\eta_1\left[\mathcal{H}\delta\eta_1{}' + \left(\mathcal{H}' + 2\mathcal{H}^2\right)\delta\eta_1 - 2\psi_1' - 4\mathcal{H}\psi_1\right] - \mathcal{H}\delta\eta_2$$

and the field fluctuation as

$$\widetilde{\delta\varphi_2} = \delta\varphi_2 + \varphi_0'\delta\eta_2 + \delta\eta_1\left[\varphi_0''\delta\eta_1 + \varphi_0'\delta\eta_1{}' + 2\delta\varphi_1{}'\right]$$

- Can construct field fluctuations on uniform curvature hypersurfaces, or Sasaki-Mukhanov variables,

$$\begin{aligned} \mathcal{Q}_2 = {} & \delta\varphi_2 + \frac{\varphi_0'}{\mathcal{H}}\psi_2 + \left(\frac{\psi_1}{\mathcal{H}}\right)^2\left[2\mathcal{H}\varphi_0' + \varphi_0'' - \frac{\mathcal{H}'}{\mathcal{H}}\varphi_0'\right] \\ & + 2\frac{\varphi_0'}{\mathcal{H}^2}\psi_1'\psi_1 + \frac{2}{\mathcal{H}}\psi_1\delta\varphi_1 \end{aligned}$$

M. and Wands (2003)

Second order scalar field dynamics

- perturbing $\nabla_\mu T^{\mu\nu}(\varphi) = 0$ to second order
- using the Einstein equations at 0th, 1st, and 2nd order
- Klein-Gordon equation 0th, and 1st order,

get the second order Klein Gordon equation in terms of the Sasaki-Mukhanov variables on large scales

$$\begin{aligned}
\mathcal{Q}_2{}'' \quad &+ \quad 2\mathcal{H}\mathcal{Q}_2{}' + a^2\left[U_{,\varphi\varphi} + \frac{16\pi G}{\mathcal{H}} U_0 \varphi_0' \left(\frac{U_{,\varphi}}{U_0} + \frac{4\pi G}{\mathcal{H}}\varphi_0'\right)\right]\mathcal{Q}_2 \\
&+ \quad a^2 \left\{ U_{,\varphi\varphi\varphi} + \frac{24\pi G}{\mathcal{H}}\varphi_0' U_{,\varphi\varphi} + \frac{8\pi G}{\mathcal{H}}\left[3U_{,\varphi}\left(\frac{8\pi G}{\mathcal{H}}\left(\varphi_0'{}^2 - 2a^2 U_0\right) - a^2\frac{U_{,\varphi}}{\varphi_0'}\right)\right.\right. \\
&\qquad\qquad \left.\left. + \left(\frac{8\pi G}{\mathcal{H}}\right)^2 U_0\varphi_0'\left(\varphi_0'{}^2 - 3a^2 U_0\right)\right]\right\}\mathcal{Q}_1^2 = 0
\end{aligned}$$

M. 2005

Note source terms quadratic in $\mathcal{Q}_1$

How they are related. . .

Relating the field fluctuation to ζ_2

$$\zeta_2 = -\frac{\mathcal{H}}{\varphi_0'}\mathcal{Q}_2 - \left[7 - 3c_{\rm s}^2 - 3\frac{{\varphi_0'}^2 - a^2U_0}{a^2\rho_0} + 3\frac{a^2U_{,\varphi}}{\mathcal{H}\varphi_0'}\right]\left(\frac{\mathcal{H}}{\varphi_0'}\right)^2\mathcal{Q}_1^2$$

M. 2005

where ζ_2 is the curvature perturbation on uniform density hypersurfaces

Note:

- Only approximation so far: large scales, no slow roll approximation
- no non-local terms, like e.g. $\nabla^{-2}(\nabla\delta X\nabla\delta X)$

Multi-field case at second order

- Defining gauge-invariant variables for a particular field, like $\mathcal{Q}_{2I}$, no problem
- Using slow roll
  - scalar field dynamics, i.e. Klein-Gordon equation
  - relating variables, i.e. $\zeta_2 = \zeta_2(\mathcal{Q}_{2I})$

    $\Rightarrow$ no problem
  - agreement with ΔN formalism

M. 2005

- *But*: not using slow roll is problematic

Wazza problem?

- The problem: $0-i$ Einstein equation at second order

$$\mathcal{H}\left(\widetilde{\phi_2}_{,i}-4\widetilde{\phi_1}\widetilde{\phi_1}_{,i}\right)-4\pi G\sum_K\left(\varphi'_{0K}\mathcal{Q}_{2K,i}+\ 2\mathcal{Q}_{1K}{}'\mathcal{Q}_{1K,i}\ \right)+O(k^3)=0$$

Wazza problem?

- The problem: $0-i$ Einstein equation at second order

$$\mathcal{H}\left(\widetilde{\phi_2}_{,i}-4\widetilde{\phi_1}\widetilde{\phi_1}_{,i}\right)-4\pi G\sum_K\left(\varphi'_{0K}\mathcal{Q}_{2K,i}+\boxed{2\mathcal{Q}_{1K}{}'\mathcal{Q}_{1K,i}}\right)+O(k^3)=0$$

- Different ways to deal with this term, and the gradients:
  - slow roll: replace $\mathcal{Q}_{1K}{}'$ using first order Klein-Gordon equation $\Rightarrow$ term becomes $\propto(\mathcal{Q}_{1K}{}^2)_{,i}$ or
  - take divergence of $0-i$ Einstein equation $\Rightarrow$ introduces non-local terms

Acquaviva, Bartolo, Matarrese and Riotto 2003

- Problem doesn't arise in single field case, because constraint equation

$$\sum_K\varphi'_{0K}\mathcal{Q}_{1K}{}'=-a^2\sum_K\left(U_{,\varphi_K}+\frac{8\pi G}{\mathcal{H}}U_0\varphi'_{0K}\right)\mathcal{Q}_{1K}$$

reduces in single field case to $\mathcal{Q}'_1\propto\mathcal{Q}_1$

Conclusions

- Second order cosmological perturbation theory studying scalar field dynamics (so far on large scales) works nicely for
  - single field case, and
  - multi-field slow roll case
- Details in: astro-ph/0506532v2

Outlook

work in progress:

- include small scales, non-local term “issue” should get clearer if small scales are included
- no slow-roll approximation
- include vector and tensor perturbations