A New Formalism for the Spectrum of Inflationary Curvature Perturbations

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Outline

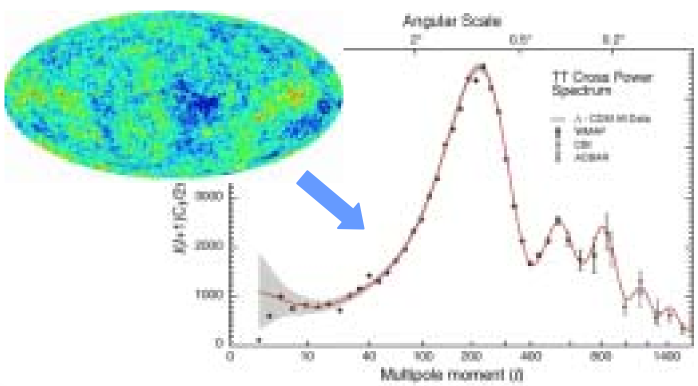
- Introduction
- Wronskian Formalism
- Extended General Slow-roll Formula
  - Long wavelength approx. and General slow-roll approx.
  - Matching two approximations
- Summary

1. Introduction - I -

● Precision Cosmology - Observational side -

Thanks to recent advances in observational technologies,
We can determine, or constrain possible models / theories of the early universe.

- the first year WMAP data (2003)



- The universe is flat.
- The spectrum of the primordial perturbation is almost scale-invariant.

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Standard Inflation - Big Bang model is supported.

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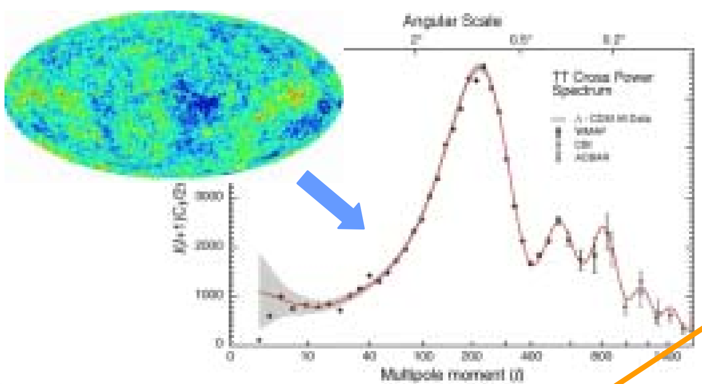

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……… **and more**

Standard Inflation - Big Bang model is supported.

Very useful tool to test various models of inflation

We need the theoretical prediction for the spectrum of curvature perturbations !

1. Introduction -II-

●Particle Cosmology - Theoretical side -

SUSY / string theories ⟶ Many scalar fields (dilaton, moduli, ……)

⟶ Various models of multi-component inflation are proposed.

It is important to extend the formula for the spectrum of the curvature perturbations to multi-component inflation.

$$\mathcal{P}_{\mathcal{R}_c}(k) = \frac{k^3}{2\pi^2}\left|\mathcal{R}_c\right|^2$$

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$$\mathcal{P}_{\mathcal{R}_c}(k) = \frac{k^3}{2\pi^2}\left|\mathcal{R}_c\right|^2$$

✧However, for multi-component inflation,
the formulation is more complicated in comparison with single-component.

In the D-component scalar field inflation,
the perturbation equations are D coupled second order differential equations.

$$\left\{\frac{d^2}{dt^2} + A(t)\frac{d}{dt} + B(t) + k^2\right\}\delta\phi^a = C^a{}_b(t)\delta\phi^b$$

It is a heavy task to solve the full set of these perturbation equations.

2. Wronskian Formalism - I -

● Basic idea

◆What we need is only the final value of the curvature perturbation on comoving hypersurfaces, $\mathcal{R}_{c\mathrm{fin}}$, on the super-horizon scales.

◆We do not need to know the evolution of all components of the multi-component field.

◆If we can identify the part of the perturbations that contributes to $\mathcal{R}_{c\mathrm{fin}}$,
we may solve only that part without solving the full set of perturbation equations.

➡ **We propose a formulation which is obtained with use of a Wronskian.**

2. Wronskian Formalism - II -

● Basic equations - D-component scalar field -

●background equations

$$3\mathcal{H}^2 \equiv 3(a'/a)^2 = \frac{1}{2}\phi'^2 + a^2 V(\phi) \qquad ' \equiv d/d\eta$$

conformal time

$$\phi^{a\prime\prime} + 2\mathcal{H}\phi^{a\prime} + a^2 V_{,\phi^a} = 0 \qquad V_{,\phi^a} \equiv \partial V/\partial\phi^a$$

●perturbations

We define a gauge invariant variable; $\delta\phi^a_{\mathrm{F}} \equiv \delta\phi^a - \frac{\phi^{a\prime}}{\mathcal{H}}\mathcal{R}$

intrinsic curvature perturbation

scalar field perturbation

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intrinsic curvature perturbation

the scalar field perturbation on the flat slicing

scalar field perturbation

The evolution equation for $\delta\phi^a_{\mathrm{F}}$ is given by

$$\varphi^a \equiv a\delta\phi^a_{\mathrm{F}}$$

$$\boxed{\left\{(d/d\eta)^2 - (\mathcal{H}^2 + \mathcal{H}') + k^2\right\}\varphi^a = \left\{\frac{1}{a^2}\left(\frac{a^2\phi^{a\prime}\phi^{c\prime}}{\mathcal{H}}\right)'\delta_{cb} - a^2 V^{,\phi^a}{}_{,\phi^b}\right\}\varphi^b} \quad \text{- (\#)}$$

2. Wronskian Formalism -III-

Wronskian

The equation, (#) is an equation which formally takes the form of

$$\left\{ \left(\frac{d}{d\eta}\right)^2 + Q(\eta) + k^2 \right\} \varphi^a = P^a{}_b(\eta)\varphi^b \quad \textbf{- (\#\#)}$$

We introduce a Wronskian as

$$\boxed{W(\varphi, n) \equiv \varphi \cdot n' - \varphi' \cdot n}$$

$$\varphi \cdot n \equiv \delta_{ab}\varphi^a n^b$$

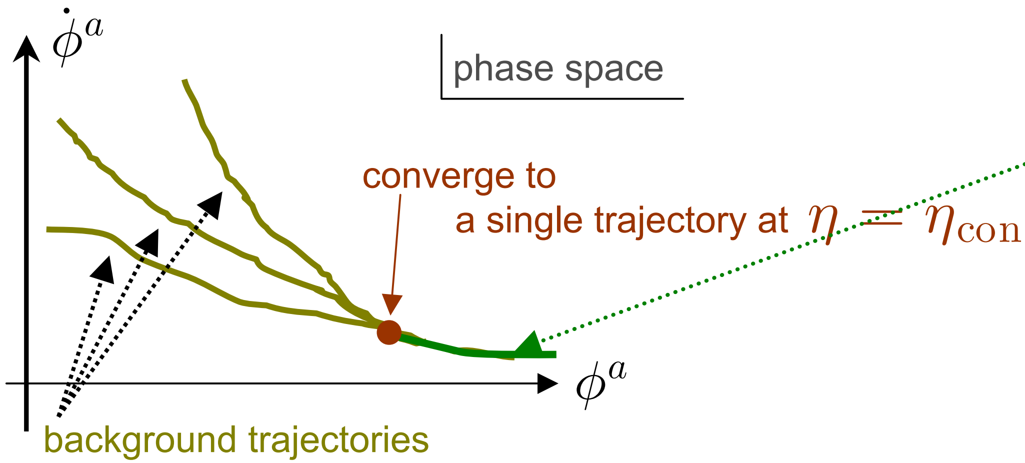
which has the property;

$$\boxed{W(\varphi, n) = \text{const.}}$$

if φ, n are solutions of the perturbation equation, **(##)** .

2. Wronskian Formalism -IV-

Defining the boundary conditions



At $\eta > \eta_{\rm con}$,

$\mathcal{R}_c \simeq \mathrm{const.}$

on super-horizon scales.

For multi-component field,
$$\boxed{\mathcal{R}_c = -\frac{\mathcal{H}}{a\phi'^2}\phi' \cdot \varphi}$$

In comoving gauge,

$$\phi' \cdot \delta\phi = 0$$

By defining n^a so that it satisfies
the boundary conditions at $\eta = \eta_{\rm con}$;

$$\boxed{\frac{dn^a}{d\eta} = -\frac{\mathcal{H}}{a\phi'^2}\phi^{a\prime}, \qquad n^a = 0}$$

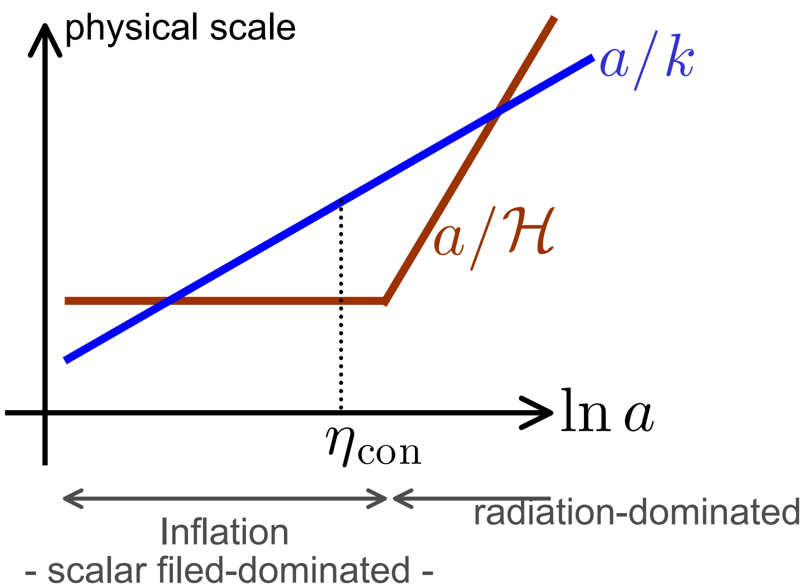
At $\eta = \eta_{\rm con}$, the Wronskian agree with

$$\boxed{W(\varphi, n) = \mathcal{R}_c(\eta_{\rm con}) \simeq \mathcal{R}_{c{\rm fin}}}$$

2. Wronskian Formalism - V -

- **The advantage of this formulation**

  - $W(\varphi, n) = \text{const.}$ → This Wronskian can be evaluated at any time.

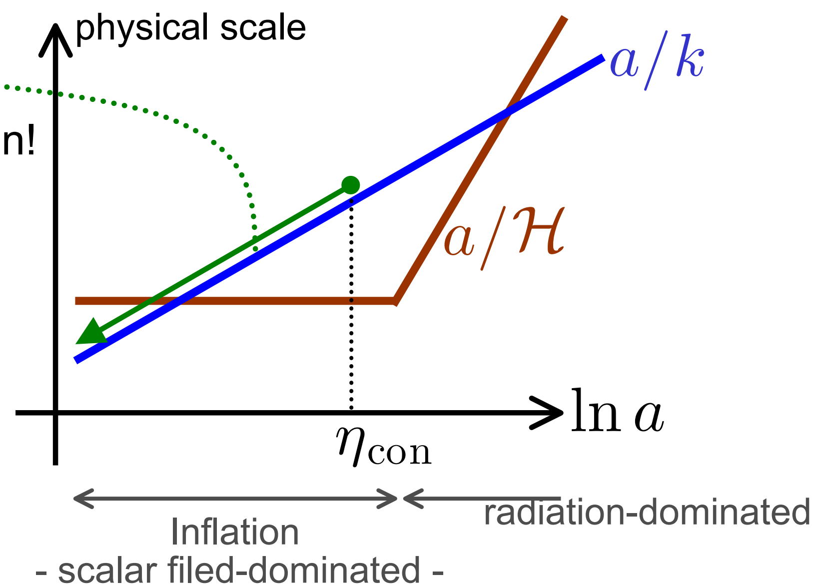


2. Wronskian Formalism - V -

The advantage of this formulation

- $W(\varphi, n) = \mathrm{const.}$ ⟶ This Wronskian can be evaluated at any time.

Once n^a is solved backward in time
with the boundary conditions,
we don't need to solve φ^a in the forward direction!



- φ^a ⟷ To solve all 2D independent modes
- n^a ⟷ To solve a single mode !!

✧ This formulation is a powerful tool for the models with extradimensions. ($D \to \infty$)
(Ref. Kobayashi and Tanaka (2005))

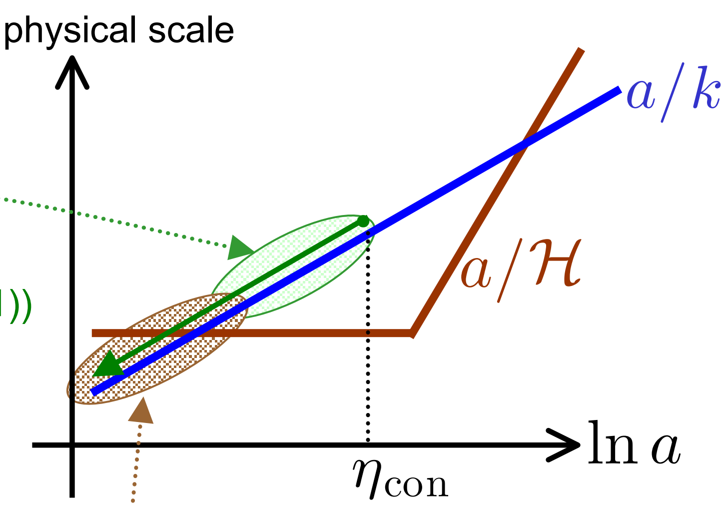
3. Extended general slow-roll formula

- I -

Long wavelength approximation

- We still need to solve n^a backward for each k^2 .

long wavelength approximation (Leach, et al. (2001))



General slow-roll approximation

- Long wavelength approx. does not hold
  when the wavelength becomes shorter than horizon scales.

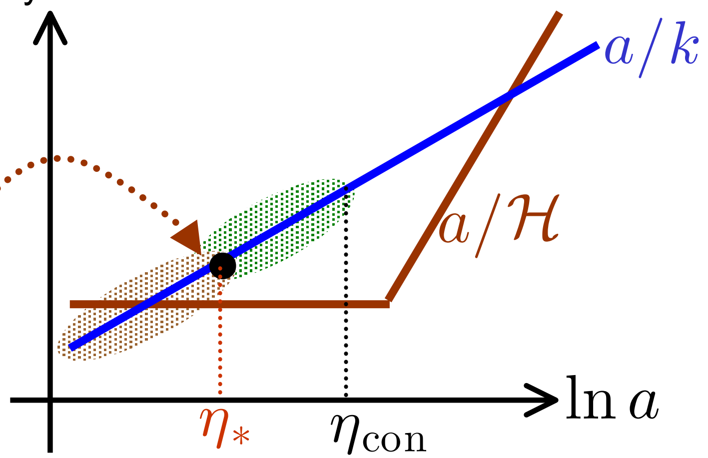
general slow-roll approximation (Stewart (2002))

By using these approximations, we can evaluate
the evolution of mode function to a large extent to analytically.

3. Extended general slow-roll formula

-II-

● Matching two approximations



- We think of matching these two approx. at an appropriate time $\eta = \eta_*$.

  The mode is already well outside the horizon scale but the general slow-roll approx. still hold at this time.

- However, the matching doesn't seem to be so trivial.

- We consider the matching by using the constancy of the Wronskian.

$$W(\varphi, n) = \text{const.}$$
$$W(\varphi, n) = \mathcal{R}_c(\eta_{\rm con}) \simeq \mathcal{R}_{c{\rm fin}}$$

$$\Rightarrow \boxed{W(\varphi, n)(\eta_*) \simeq \mathcal{R}_{c{\rm fin}}}$$

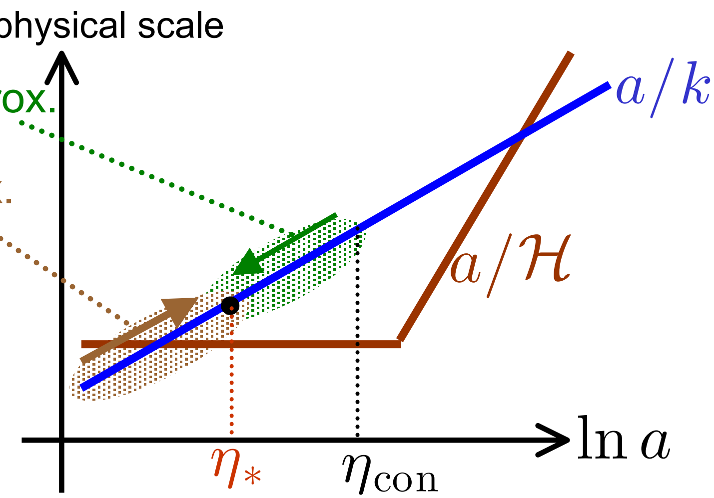
- If we know both φ^a and n^a at $\eta = \eta_*$, what we need to do is simply to compute the Wronskian, W there.

3. *Extended general slow-roll formula* -III-

- To obtain φ^a and n^a at $\eta = \eta_*$,

n^a: in the backward direction by using long wavelength approx.

φ^a: in the forward direction by using general slow-roll approx.



- n^a
  - In long wavelength approx., we can find solution for n^a in the series expansion with respect to k^2 as

$$n^a = n^{a(0)} + \delta n^{a(1)} + \cdots + \delta n^{a(q)} + \cdots$$

$$\delta n^{a(q)} = \mathcal{O}(k^{2q})$$

We also introduce

$$\boxed{\begin{aligned} & n^{a(q)} \equiv n^{a(0)} + \sum_{i=1}^{q} \delta n^{a(i)}, \\ & \left\{ \left(\frac{d^2}{d\eta^2} + Q(\eta) + k^2 \right) \delta^a{}_b - P^a{}_b \right\} n^{b(q)} = k^2 \delta n^{b(q)} \end{aligned}}$$

3. Extended general slow-roll formula

-IV-

- φ^a

- In the general slow-roll approx., we consider that $P^a{}_b$ is small.

$$\left\{\left(\frac{d}{d\eta}\right)^2 + Q(\eta) + k^2\right\}\varphi^a = P^a{}_b(\eta)\varphi^b \quad \textbf{- (\#\#)}$$

In an expansion with respect to $P^a{}_b$, we can write φ^a as

$$\varphi^a = \varphi^a_0 + \Delta\varphi^a_1 + \cdots + \Delta\varphi^a_p + \cdots$$

$$\Delta\varphi^a_p = \mathcal{O}(\mathbf{P}^p)$$

$$\mathbf{P} \equiv P^a{}_b$$

We also introduce

$$\boxed{\begin{aligned}&\varphi^a_p \equiv \varphi^a_0 + \sum_{i=1}^{p}\Delta\varphi^a_i,\\ &\left\{\left(\frac{d^2}{d\eta^2} + Q(\eta) + k^2\right)\delta^a{}_b - P^a{}_b\right\}\varphi^b_p = -P^a{}_b\Delta\varphi^b_p\end{aligned}}$$

The first order correction is obtained as

$$\boxed{\begin{aligned}&\Delta\varphi^a(\eta) = \underline{\Pi}(\eta)\left[\int_{-\infty}^{\eta_*} d\eta' P^a_b(\eta')\varphi^b_0(\eta')\left\{\underline{u_0}(\eta)u^*_0(\eta') - u_0(\eta')u^*_0(\eta)\right\}\right]\\ &\Pi(\eta) \equiv W(u^*_0, u_0)^{-1}, \quad \left(\frac{d^2}{d\eta^2} + Q(\eta) + k^2\right)u_0 = 0\end{aligned}}$$

3. Extended general slow-roll formula

-V-

● Results ~ Power Spectrum ~

To the first order in $\mathbf{P}$ and to the q-th order in k^2, $\langle|\mathcal{R}_{c\mathrm{fin}}|^2\rangle$ be evaluated as

$$\langle|\mathcal{R}_{c\mathrm{fin}}|^2\rangle_1^{(q)} = \Bigg[|\mathbf{\Phi}|^2 + 4i\Pi\int_{-\infty}^{\eta_*} d\eta' \Bigg\{ \Big(\mathrm{Im}[u_0^*(\eta')\mathbf{\Phi}]\cdot\mathbf{P}(\eta')\mathrm{Re}[u_0^*(\eta')\mathbf{\Phi}]\Big) - \theta(\eta-\eta_k)\Big(\mathrm{Im}[u_0^*(\eta')\mathbf{\Phi}]\cdot\mathbf{P}(\eta')\mathrm{Re}[u_0^*(\eta')\mathbf{\Phi}]\Big)\Bigg\}\Bigg]^{(q)}$$

$$\mathbf{\Phi} \equiv \big[W(u_0,\mathbf{n})\big]_{\eta_k}\ ,\ \mathbf{n} \equiv n^a$$

3. Extended general slow-roll formula

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● Results ～ Power Spectrum ～

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Scale invariant + higher order corrections in k^2

the zero-th order in k^2 and $\mathbf{P}$

first order correction in $\mathbf{P}$

✧The extension to the higher order in $\mathbf{P}$ is rather straight forward in our formulation.

4. Summary

- We have proposed a new formulation for systematic derivation of formulas for the spectrum of curvature perturbation from multi-component inflation.

  - We proposed the formulation using the Wronskian which is constant in time.
    Using this formulation, we can evaluate the final value of curvature perturbations without solving the full set of perturbation equations.

  - As an example to show the efficiency of this new approach, we have shown an extended general slow-roll formula.
    In this formula, we took into account the merit of the long wavelength approximation.

In the past work (ex. Stewart(2002), Cho et al. (2004)),
the general slow-roll conditions are assumed to be satisfied until the time when inflation ends,
while here it is assumed to be so until the time shortly after the horizon crossing time.