Quantization of scalar perturbations in brane-world inflation

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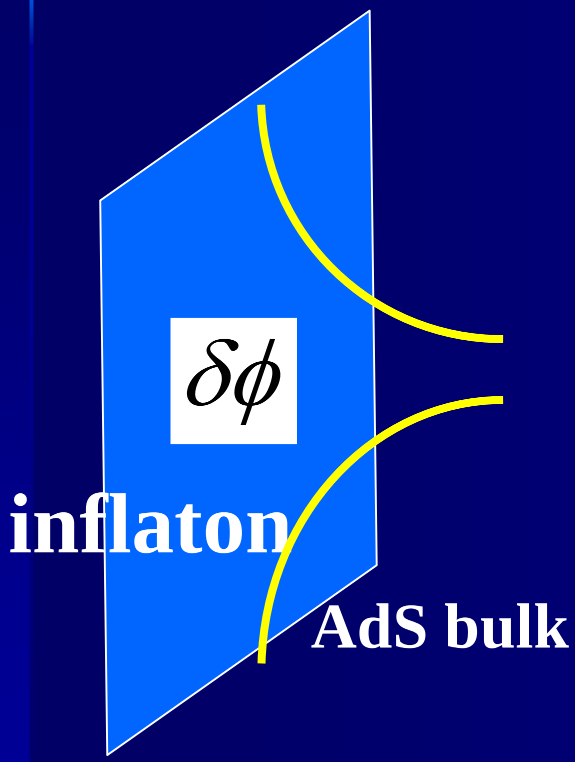
Ref. HY and K. Koyama, PRD 71 043519 (2005)

1. Introduction

We want to determine the initial condition of cosmological perturbations in brane-world inflation

Randall-Sundrum model + 4D scalar field

Maartens, Wands, Bassett, Heard



amplitude of scalar perturbations

$$\left\langle Q^2 \right\rangle = \left(\frac{H}{2\pi}\right)^2, \quad Q = \delta\phi - \frac{\dot{\phi}}{H}R$$

Mukhanov-Sasaki variable

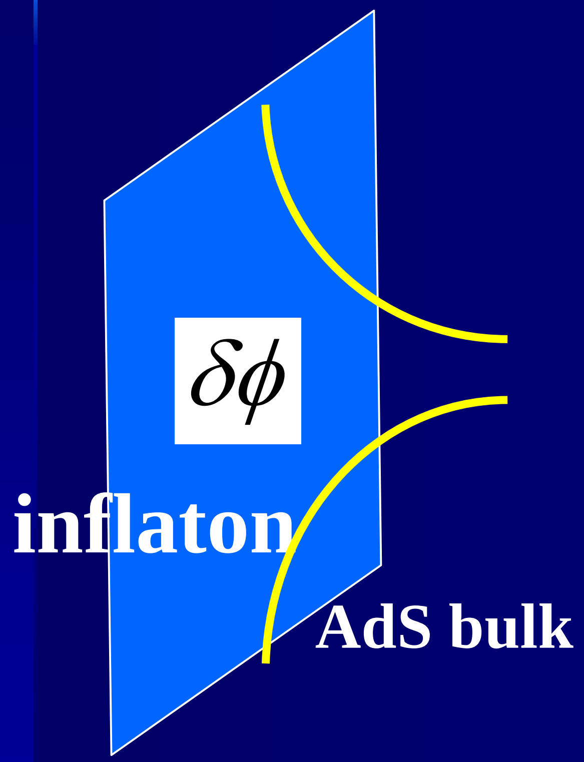
in extreme slow-roll limit ($\varepsilon \to 0$)

(no coupling between inflaton and bulk perturbations)

However, at first order in ε

bulk perturbations → scalar field fluctuation correction

Koyama, Langlois, Maartens, Wands



To compute amplitude of Q at this order

Quantum theory of bulk gravitational field as well as inflaton field on the brane

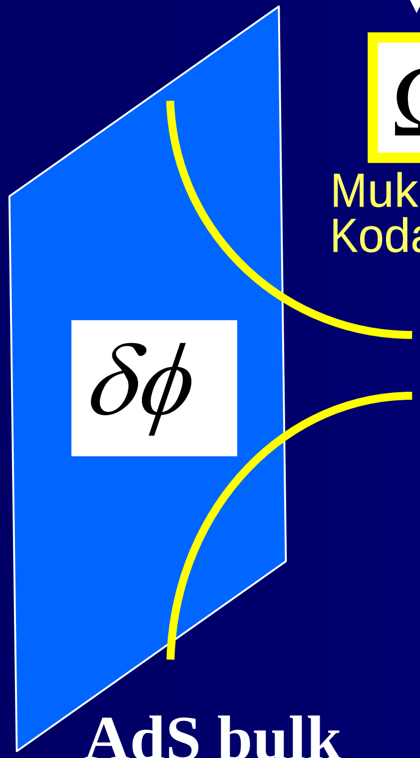
For 5D vacuum spacetime

…physical degree of freedom for scalar perturbation = 1

$\downarrow$

$\boxed{\Omega}$ **Master variable in AdS**

Mukohyama
Kodama, Ishibashi, Seto



AdS bulk

To perform quantization of bulk perturbations, we need action of Ω.

Contents of this talk

- **action of Ω with vacuum brane**
- **action of Q coupled to Ω**

2. Background

Set up (RS brane-world)

$$S=\int d^5x\sqrt{-g_5}\left[\frac{1}{2\kappa^2}\left({}^{(5)}R+12\mu^2\right)\right]-\int d^4x\sqrt{-g_4}\left(\sigma+L_\phi-\kappa^2K\right)$$

$\mu(=1/l)$ the curvature scale of AdS

$\kappa^2\sigma = 6\mu$ the tension of the brane

$$\boxed{L_\phi=V_0-\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi+V(\phi)}$$

where

$V_0 >> V(\phi)$

In zero-th order of ε , spacetime of the brane is de Sitter.

K trace of the extrinsic curvature of the brane

Background

de Sitter brane in AdS bulk

$$ds^2 = e^{2W(z)}\left(-dt^2 + e^{2\alpha(t)}\delta_{ij}dx^i dx^j + dz^2\right)$$

conformal bulk-coordinate

$$e^{\alpha(t)} = e^{Ht}$$

scale factor

$$e^{W(z)} = \left(\frac{\sinh Hz}{\sinh Hz_0}\right)^{-1}$$

z_0 location of the brane

3. Master variable

Introduction of Ω

Perturbed metric (5D longitudinal gauge)

$$ds^2 = e^{2W(z)}\left(-(1+2\underline{\Phi})dt^2 + e^{2\alpha(t)}(1+2\underline{\Psi})\delta_{ij}dx^i dx^j + 2\underline{S}dtdz + (1+2\underline{N})dz^2\right)$$

These metric variables satisfy the three constraint equations which are parts of the Einstein equations.

$$N + \Phi + \Psi = 0,$$

$$-\Phi' - 2\Psi' + 3W'N - \frac{1}{2}\left(\dot{S} + HS\right) = 0,$$

$$-\dot{N} + HN - 2\dot{\Psi} + 2H\Phi + \frac{1}{2}\left(S' + 3W'S\right) = 0.$$

3. Master variable

These constraints are satisfied if we write the metric variables using the master variable as follows.

$$\Phi = -\frac{e^{-\alpha-3W}}{6}\left(2\Omega'' - 3W'\Omega' + \ddot{\Omega} - \mu^2 e^{2W}\Omega\right)$$

$$S = e^{-\alpha-3W}\left(\dot{\Omega}' - W'\dot{\Omega}\right)$$

$$N = \frac{e^{-\alpha-3W}}{6}\left(\Omega'' - 3W'\Omega' + 2\ddot{\Omega} + \mu^2 e^{2W}\Omega\right)$$

$$\Psi = \frac{e^{-\alpha-3W}}{6}\left(\Omega'' - \ddot{\Omega} - 2\mu^2 e^{2W}\Omega\right)$$

Using these expressions, we can show that the remaining parts of the Einstein eq. are equivalent to

$$\Delta''_{(S)} - W'\Delta'_{(S)} - \mu^2 e^{2W}\Delta_{(S)} = 0,$$

$$\ddot{\Delta}_{(S)} - W'\Delta'_{(S)} + \mu^2 e^{2W}\Delta_{(S)} = 0,$$

$$\dot{\Delta}'_{(S)} - W'\dot{\Delta}_{(S)} = 0$$

where

$$\Delta\Omega = \delta^{ij}\Omega_{,ij}$$

$$\Delta_{(S)} = e^{2\alpha}\left[\ddot{\Omega} - 3H\dot{\Omega} - \left(\Omega'' - 3W'\Omega'\right) - \mu^2 e^{2W}\Omega - e^{-2\alpha}\Delta\Omega\right]$$

3. Master variable

A replacement of Ω with $\tilde{\Omega}$ does not alter the metric variables.

$$\Phi = -\frac{e^{-\alpha-3W}}{6}\left(2\Omega'' - 3W'\Omega' + \ddot{\Omega} - \mu^2 e^{2W}\Omega\right)$$

$$S = e^{-\alpha-3W}\left(\dot{\Omega}' - W'\dot{\Omega}\right)$$

$$N = \frac{e^{-\alpha-3W}}{6}\left(\Omega'' - 3W'\Omega' + 2\ddot{\Omega} + \mu^2 e^{2W}\Omega\right)$$

$$\Psi = \frac{e^{-\alpha-3W}}{6}\left(\Omega'' - \ddot{\Omega} - 2\mu^2 e^{2W}\Omega\right)$$

$$\tilde{\Omega} = \Omega - \frac{1}{k^2}\Delta_{(S)},$$

$$\Delta_{(S)} = e^{2\alpha}\left[\ddot{\Omega} - 3H\dot{\Omega} - \left(\Omega'' - 3W'\Omega'\right) - \mu^2 e^{2W}\Omega - e^{-2\alpha}\Delta\Omega\right]$$

And also, we can set $\Delta_{(S)} = 0$ by this replacement.

satisfy the perturbed Einstein equations.

In summary,

solution

$$\Phi = -\frac{e^{-\alpha-3W}}{6}\left(2\Omega'' - 3W'\Omega' + \ddot{\Omega} - \mu^2 e^{2W}\Omega\right)$$

$$S = e^{-\alpha-3W}\left(\dot{\Omega}' - W'\dot{\Omega}\right)$$

$$N = \frac{e^{-\alpha-3W}}{6}\left(\Omega'' - 3W'\Omega' + 2\ddot{\Omega} + \mu^2 e^{2W}\Omega\right)$$

$$\Psi = \frac{e^{-\alpha-3W}}{6}\left(\Omega'' - \ddot{\Omega} - 2\mu^2 e^{2W}\Omega\right)$$

where Ω is a solution of the master eq.

$$\ddot{\Omega} - 3H\dot{\Omega} - \left(\Omega'' - 3W'\Omega'\right) - e^{2\alpha}\Delta\Omega - \mu^2 e^{2W}\Omega = 0$$

$$\left(\Delta_{(S)} = 0 \right)$$

3. Master variable

General Solution $$\Omega = \int d^3k \, dm \, v_m(t) \, u_m(z) \, e^{i\vec{k}\cdot\vec{x}}$$

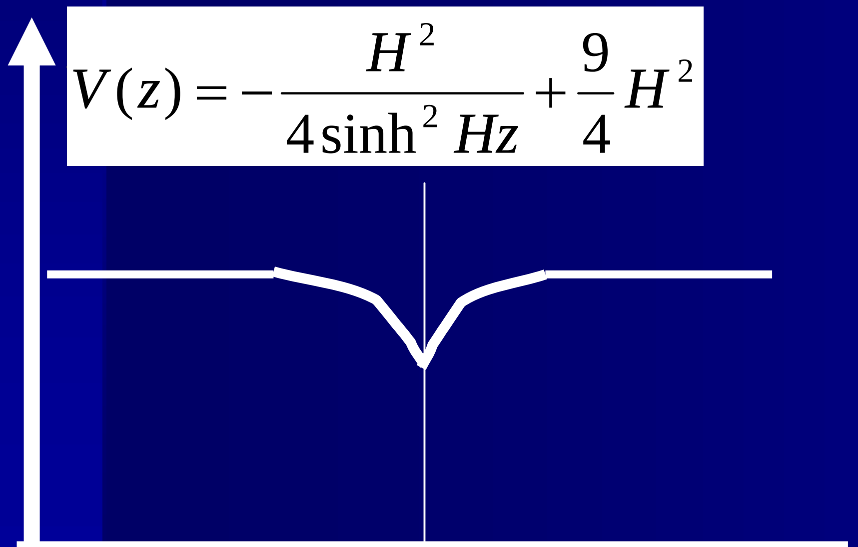
$$\ddot{v}_m - 3H\dot{v}_m + \left[m^2 + k^2 a^{-2}\right] v_m = 0$$

$$u_m'' - 3W' u_m' + \left[m^2 + \mu^2 e^{2W}\right] u_m = 0$$

$$v_m = \frac{\sqrt{-k\eta}}{(-H\eta)^2} Z_\nu(-k\eta), \quad \nu^2 = \frac{9}{4} - \frac{m^2}{H^2}$$ **Bessel**

$$u_m = (\sinh Hz)^{-1} W_{\nu-1/2}(\cosh Hz)$$ **Legendre**

$$V(z) = -\frac{H^2}{4\sinh^2 Hz} + \frac{9}{4}H^2$$



brane

$$m > \frac{3}{2}H$$ **KK continuum**

$$m < \frac{3}{2}H$$ **Non-normalizable (for vacuum brane)**

second order action

$$S=\int d^5x\sqrt{-g_5}\left[\frac{1}{2\kappa^2}\left({}^{(5)}R+12\mu^2\right)\right]-\int d^4x\sqrt{-g_4}\left(\sigma+V_0-\kappa^2K\right)$$

$$\delta_2 S=\int d^5x\frac{e^{-3\alpha-3W}}{6\kappa^2}\left[\left(\Delta\dot{\tilde{\Omega}}\right)^2-\left(\Delta\tilde{\Omega}'\right)^2+e^{-2\alpha}\Delta\tilde{\Omega}\Delta^2\tilde{\Omega}+\mu^2e^{2W}\left(\Delta\tilde{\Omega}\right)^2\right]$$
$$+\int d^4x\frac{e^{\alpha}}{6\kappa^2}\left[\frac{9}{2}W'\dot{F}^2-\frac{3}{2}W'e^{-2\alpha}F\Delta F-W'e^{-4\alpha}\left(\Delta\tilde{\Omega}\right)^2-3e^{-2\alpha}\ddot{F}\Delta\left(\Omega-\tilde{\Omega}\right)\right]$$

$$F\equiv\Omega'-W'\Omega$$

$$\tilde{\Omega}=\Omega-\frac{1}{k^2}\Delta_{(S)},$$
$$\Delta_{(S)}=e^{2\alpha}\left[\ddot{\Omega}-3H\dot{\Omega}-\left(\Omega''-3W'\Omega'\right)-\mu^2e^{2W}\Omega-e^{-2\alpha}\Delta\Omega\right]$$

This action contains higher derivative terms (time).

quantization $\longrightarrow$ up to second derivatives

$\longrightarrow$ set $\Delta_{(S)}=0$

second order action

$$\boxed{S=\int d^5x\,\frac{e^{-3\alpha-3W}}{6\kappa^2}\left[\left(\Delta\dot{\Omega}\right)^2-\left(\Delta\Omega'\right)^2+e^{-2\alpha}\Delta\Omega\Delta^2\Omega+\mu^2e^{2W}\left(\Delta\Omega\right)^2\right]+\int d^4x\,\frac{e^{\alpha}}{6\kappa^2}\left[\frac{9}{2}W'\dot{F}^2-\frac{3}{2}W'e^{-2\alpha}F\Delta F+W'e^{-4\alpha}\left(\Delta\Omega\right)^2\right]}$$

variation with respect to Ω

$$\boxed{F\equiv\Omega'-W'\Omega}$$

$$\left\{\begin{array}{l}\boxed{\ddot{\Omega}-3H\dot{\Omega}-\left(\Omega''-3W'\Omega'\right)-e^{2\alpha}\Delta\Omega-\mu^2e^{2W}\Omega=0}\\ \boxed{F=0}\quad\text{junction condition}\end{array}\right\}$$

This is consistent with the eq. obtained by variation with respect to F.

4. Quantum graviton with Vacuum brane

4.Quantum graviton with vacuum brane

quantization of heavy modes (m>3H/2)

$\boxed{F=0}$ junction condition

$$u_m(z)=C(m)(\sinh Hz)^{-1}\left(P_{i\gamma-1/2}(\cosh Hz)+\beta(m)Q_{i\gamma-1/2}(\cosh Hz)\right),$$

$$\gamma=\sqrt{\frac{m^2}{H^2}-\frac{9}{4}},\quad \beta(m)=-\frac{P^1{}_{i\gamma-1/2}(\cosh Hz_0)}{Q^1{}_{i\gamma-1/2}(\cosh Hz_0)}$$

C(m) is determined by the normalization condition,

$$2\int_{z_0}^{\infty}d(Hz)\,e^{-3W}u_m(z)u^*_{m'}(z)=\delta(\gamma'-\gamma)$$

as

$$C(m)=\left(\frac{H}{\mu}\right)^{3/2}\frac{1}{\sqrt{\varsigma(m)+\xi(m)}},$$

$$\varsigma(m)=\left|\frac{\Gamma(i\gamma)}{\Gamma(i\gamma+1/2)}\right|^2,\ \xi(m)=\left|\frac{\Gamma(-i\gamma)}{\Gamma(-i\gamma+1/2)}+\pi\beta(m)\frac{\Gamma(i\gamma+1/2)}{\Gamma(i\gamma+1)}\right|^2$$

4.Quantum graviton with vacuum brane

The time mode function is normalized so that the correct canonical quantization is ensured.

$$v_m(z) = \frac{\sqrt{3\kappa^2}}{-k^2}\frac{\sqrt{\pi}}{2}(-H\eta)^{-3/2}e^{-\gamma\pi/2}H_{i\gamma}^{(1)}(-k\eta)$$

The energy density of $\delta E_{\mu\nu}$ **is given in terms of the master variable by**

$$\kappa_4^2\delta\rho_E = \frac{k^4 e^{-5\alpha}}{3}\Omega$$

We define its power spectrum $P_E(k)$ normalized by ρ_Λ**.**

$$\frac{\langle\delta\rho_E^2\rangle}{\rho_\Lambda^2} = \int\frac{dk}{k}P_E(k)$$

$$\boxed{P_E(k) = \frac{1}{108\pi^2}\left(\frac{k\,e^{-\alpha}}{H}\right)^7(\kappa_4 H)^2 C_{KK}^2}$$

The amplitude C_{KK} is enhanced For large H/μ as expected.

5. Scalar field on the brane

Equation of motion for scalar perturbations

Koyama, Langlois, Maartens, Wands

AdS bulk

Ω Master variable

$\|$

metric perturbation $\supset$ $\delta E_{\mu\nu}$

$$\delta\phi = \delta\phi_0 + \varepsilon\delta\phi_1$$

ε slow roll parameter

$$\delta\ddot{\phi}_0 + 3H\delta\dot{\phi}_0 + k^2 a^{-2}\delta\phi_0 = 0$$

$$\delta\phi_0 = \frac{C_1}{-k\eta}\left(-ik - \frac{1}{\eta}\right)e^{-ik\eta}$$

C_1 ; normalization constant

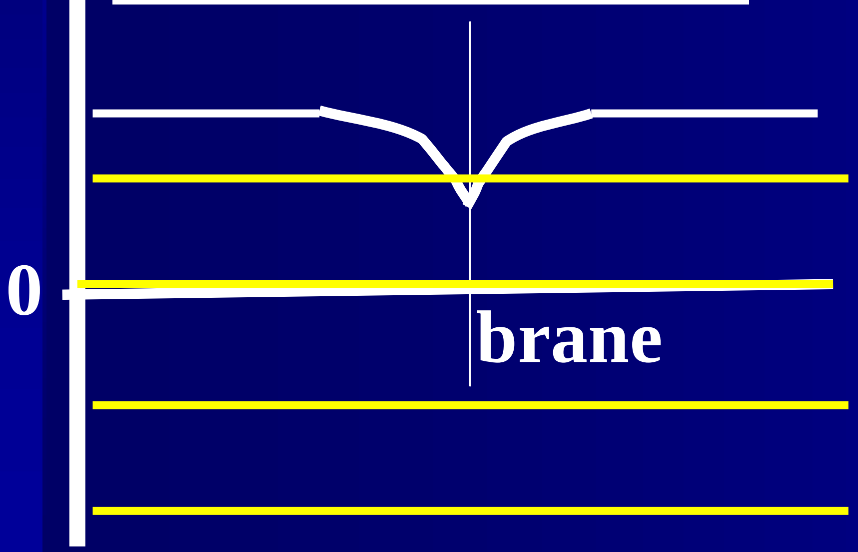
Equation of motion for scalar perturbations

Koyama, Langlois, Maartens, Wands

solution with $\boxed{\delta\phi_0}$ on the brane

$$\Omega = C_1\sqrt{2\pi}\sum_{l=0}^{\infty}(-1)^l\left(2l+\frac{1}{2}\right)\frac{(\sinh Hz)^{-1}Q_{2l}(\cosh Hz)}{\mu\, Q_{2l}^1(\cosh Hz_0)}(-k\eta)^{-3/2}J_{2l+1/2}(-k\eta)$$
$$+iC_1\sqrt{2\pi}\sum_{l=0}^{\infty}(-1)^l\left(2l+\frac{3}{2}\right)\frac{(\sinh Hz)^{-1}Q_{2l+1}(\cosh Hz)}{\mu\, Q_{2l+1}^1(\cosh Hz_0)}(-k\eta)^{-3/2}J_{2l+3/2}(-k\eta)$$

$$V(z) = -\frac{H^2}{4\sinh^2 Hz} + \frac{9}{4}H^2$$



This solution is composed of $m^2=2H^2$ mode, zero-mode, and infinite ladder of discrete tachyonic modes.
(They are normalizable in this case.)

Evolution equation of Q

$$\delta\ddot{\phi}_1 + 3H\delta\dot{\phi}_1 + k^2 a^{-2}\delta\phi_1 = -V''\delta\phi_0 - 3\dot{\phi}\dot{\Psi} + \dot{\phi}\dot{\Phi} - 2V'\Phi$$

Einstein eq. on the brane

$$\underbrace{\ddot{Q}_1 + 3H\dot{Q}_1 + k^2 a^{-2} Q_1 = -m_{eff}^2 Q_0}_{\text{4D cosmology}} + \boxed{J}$$

$$\left(Q_1 = \delta\phi_1 - \frac{\dot{\phi}}{H}\Psi \,, \quad m_{eff}^2 = V'' + 6\dot{H} \right)$$

$$J = -\frac{\dot{\phi}}{H}\frac{k^2}{6a^3}\left(\ddot{\Omega} - H\dot{\Omega} + \frac{k^2}{a^2}\Omega \right)$$

…… **bulk perturbations**

→ **To address the quantization of Q, we need the second order action for Q coupled to bulk perturbations.**

5. Scalar field on the brane

We add the action of the scalar field.

$$S=\int d^5x\sqrt{-g_5}\left(\frac{1}{2\kappa^2}R-\Lambda_5\right)-\int d^4x\sqrt{-g_4}\left(\sigma-\boxed{\frac{1}{2}\partial_\mu\phi\partial^\mu\phi-V(\phi)}\right)$$

$$S=\int d^5x\frac{e^{-3\alpha-3W}}{6\kappa^2}\left[\left(\Delta\dot{\Omega}\right)^2-\left(\Delta\Omega'\right)^2+e^{-2\alpha}\Delta\Omega\Delta^2\Omega+\mu^2e^{2W}\left(\Delta\Omega\right)^2\right]$$
$$+\int d^4x\frac{e^{\alpha}}{6\kappa^2}\left[-\frac{9}{2}W'\dot{F}^2+\frac{3}{2}W'e^{-2\alpha}F\Delta F+W'e^{-4\alpha}\left(\Delta\Omega\right)^2\right]$$
$$+\frac{1}{2}\int d^4x\,e^{3\alpha}\left[\delta\dot{\phi}^2+e^{-2\alpha}\delta\phi\Delta\delta\phi-V''\delta\phi^2+2W'F\left(2\dot{\phi}\delta\dot{\phi}-V'\delta\phi\right)\right.$$
$$\left.+\dot{\phi}\delta\dot{\phi}\left(\ddot{\Omega}-5H\dot{\Omega}+4H^2\Omega-\frac{5}{3}e^{-2\alpha}\Delta\Omega\right)+V'\delta\phi\left(\ddot{\Omega}+H\dot{\Omega}-2H^2\Omega+\frac{1}{3}e^{-2\alpha}\Delta\Omega\right)\right]$$

$$\boxed{\Omega,\ F,\ \delta\phi}$$

$$F\equiv\Omega'-W'\Omega$$

Action of Q (main result)

$$S=\int d^5x\frac{e^{-3\alpha-3W}}{6\kappa^2}\left[\left(\Delta\dot{\Omega}\right)^2-\left(\Delta\Omega'\right)^2+e^{-2\alpha}\Delta\Omega\Delta^2\Omega+\mu^2e^{2W}\left(\Delta\Omega\right)^2\right]$$
$$+\int d^4x\frac{e^{3\alpha}}{2}\left[\dot{Q}^2+e^{-2\alpha}Q\Delta Q-(V''+6\dot{H})Q^2-\frac{W'e^{-6\alpha}}{3\kappa^2}\left(\Delta\Omega\right)^2\right.$$
$$\left.+\frac{\dot{\phi}\,e^{-3\alpha}}{3H}Q\Delta\left(\ddot{\Omega}-H\dot{\Omega}-e^{-2\alpha}\Delta\Omega\right)\right]$$

$\xrightarrow{\delta Q}$

$$\ddot{Q}_1+3H\dot{Q}_1+k^2a^{-2}Q_1=-m_{eff}^2Q_1+J$$
$$\left(Q_1=\delta\phi_1-\frac{\dot{\phi}}{H}R\,,\quad m_{eff}^2=V''+6\dot{H}\right)$$

two scalar field $\boxed{Q}$ (on the brane) $\boxed{\Omega}$ (in the bulk)

They are coupled to each other on the brane.

5. Scalar field on the brane

Calculation of $\langle Q^2 \rangle$

In general, it is difficult to quantize this coupled system.

$$
\begin{aligned}
S = &\int d^5x \frac{e^{-3\alpha-3W}}{6\kappa^2}\Big[\big(\Delta\dot{\Omega}\big)^2 - \big(\Delta\Omega'\big)^2 + e^{-2\alpha}\Delta\Omega\Delta^2\Omega + \mu^2 e^{2W}\big(\Delta\Omega\big)^2\Big] \\
&+ \int d^4x \frac{e^{3\alpha}}{2}\Big[\dot{Q}^2 + e^{-2\alpha}Q\Delta Q - (V''+6\dot{H})Q^2 - \frac{W' e^{-6\alpha}}{3\kappa^2}\big(\Delta\Omega\big)^2 \\
&\qquad + \boxed{\frac{\dot{\phi}\, e^{-3\alpha}}{3H}}\, Q\Delta\big(\ddot{\Omega} - H\dot{\Omega} - e^{-2\alpha}\Delta\Omega\big)\Big]
\end{aligned}
$$

suppressed $\longrightarrow$ **solve the eqs. perturbatively**

At 0-th order, $Q_0 = \delta\phi_0$ **decouples.**

$$\boxed{C_1 = \kappa^2 \frac{i\dot{\phi}}{\sqrt{2kH}}} \qquad \delta\phi_0 = \frac{C_1}{-k\eta}\left(-ik - \frac{1}{\eta}\right)e^{-ik\eta}$$

5. Scalar field on the brane

To quantize Q_1, we have to determine including the normalization, and then solve Q_1.

$$J = -\frac{\dot{\phi}}{H}\frac{k^2}{6a^3}\left(\ddot{\Omega} - H\dot{\Omega} + \frac{k^2}{a^2}\Omega\right)$$

$$\begin{aligned}\Omega = {} & C_1\sqrt{2\pi}\sum_{l=0}^{\infty}(-1)^l\left(2l+\frac{1}{2}\right)\frac{(\sinh Hz)^{-1}Q_{2l}(\cosh Hz)}{\mu Q_{2l}^1(\cosh Hz_0)}(-k\eta)^{-3/2}J_{2l+1/2}(-k\eta) \\ & + iC_1\sqrt{2\pi}\sum_{l=0}^{\infty}(-1)^l\left(2l+\frac{3}{2}\right)\frac{(\sinh Hz)^{-1}Q_{2l+1}(\cosh Hz)}{\mu Q_{2l+1}^1(\cosh Hz_0)}(-k\eta)^{-3/2}J_{2l+3/2}(-k\eta)\end{aligned}$$

$$\longleftarrow \boxed{C_1 = \kappa^2\frac{i\dot{\phi}}{\sqrt{2kH}}}$$

The normalization of the light modes which compose the solution of Ω is determined by that of Q_0, because they are related through the junction condition.

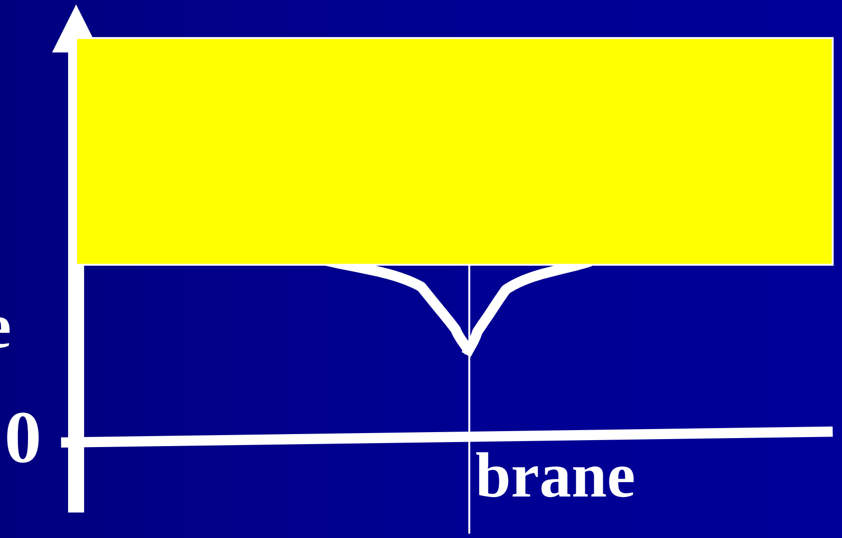
5. Scalar field on the brane

There are also contributions to Ω from heavy modes with $m>3H/2$ which satisfy vacuum boundary condition ($F=0$).

↑

action of Ω with vacuum brane (in our paper)

$$V(z) = -\frac{H^2}{4\sinh^2 Hz} + \frac{9}{4}H^2$$



From these results, the correction term J can be determined including its normalization.

Calculation of $\langle Q^2 \rangle$ using our result

⟶ future work

6. Summary

Motivation

We want to compute the amplitude of scalar pert. considering the coupling between the bulk pert. and the inflaton field.

↓

- action of Ω with a vacuum brane
- action of Q coupled to Ω

essential feature of scalar pert. in the brane world

Calculation of $\left\langle Q^2 \right\rangle$ using this action

⟶ future work

END